Name:

STAT 422 – Midterm 2

Due: Th, Dec 1, in class.

Open book and open notes. No time limit. Please do not talk with anyone except me about this exam.

[pts] indicates the points for each problem.

* indicates problems that I think might be more challenging than others.

1. [10] Chapter 5, problem 12.

2. [10] Chapter 5, problem 13.

3. [10] Let y_i be the number of misprints on page i in a certain book. Consider two alternative models:
Model H_1:

$$\begin{aligned} y_i &\sim Poi(\lambda \cdot N_i), \quad i = 1, \ldots, n \\ \lambda &\sim Ga(1, \beta) \end{aligned}$$

and model H_2:

$$\begin{aligned} y_i &\sim Bin(N_i, \theta) \\ \theta &\sim Be(1, \beta - 1), \end{aligned}$$

where N_i is the number of characters per page and $\beta > 1$ is a fixed hyperparameter.
Find the Bayes factor $B = p(y|H_2)/p(y|H_1)$ for choosing between competing models H_1 and H_2.
Hint: Neither $p(y|H_2)$ nor $p(y|H_1)$ take the form of any well-known distribution[1].

[1] Careful, $p(y|H_1)$ is *not* a Neg-bin (like on the bottom of p49). We have not just one observation, but $y_1, \ldots, y_n$.

4. We are interested in inference for a linear regression model for regressing $y_i, i = 1, \ldots, n$ on $x_{ij}, j = 1, \ldots, k$ with a proper multivariate normal prior on β and an inverse Wishart prior on the variance-covariance matrix. Let $y = (y_1, \ldots, y_n)'$ be a $n \times 1$ vector of responses, X a $n \times k$ design matrix (i.e. a matrix with x_{ij} in the i-th row and j-th column). Let $\beta = (\beta_1, \ldots, \beta_k)$ be a vector of regression coefficients.

$$\begin{aligned} y &\sim N(X\beta, \Sigma) \\ \beta &\sim N(\mu, T), \end{aligned}$$

4a. [10] Assume that Σ is known. Let $V_\beta = (X'\Sigma^{-1}X)^{-1}$, and $\hat{\beta} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$. Show that $p(\beta|\Sigma, y) = N(m, V)$ with

$$V^{-1} = T^{-1} + V_\beta^{-1} \text{ and } m = V \cdot (T^{-1}\mu + \underbrace{V_\beta^{-1}\hat{\beta}}_{X'\Sigma^{-1}y}).$$

4b*. [5] We now extend the model by assuming an unknown – but diagonal – covariance matrix:

$$\begin{aligned} y &\sim N(X\beta, \sigma^2 I) \\ \beta &\sim N(\mu, \sigma^2 R), \\ \sigma^2 &\sim \text{Inv-}\chi^2(\nu_0, s_0). \end{aligned}$$

Find $p(\lambda|y)$.

5. For the following data we will consider two competing models: H_1: a linear; H_2 : a quadratic regression.

x_i	-1.9	-0.39	0.79	-0.20	0.42	-0.35	0.67	0.63	-0.024	1.2
y_i	-1.7	-0.23	0.50	-0.66	1.97	0.10	0.60	1.13	0.943	2.6

Model H_1:

$$\begin{aligned} y_i &= \beta_1 + \beta_2 x_i + \epsilon_i, \quad i = 1, \ldots, n \\ &\epsilon_i \sim N(0,1) \\ \beta_1 &\sim N(0,1), \quad \beta_2 \sim N(1,1), \end{aligned}$$

with β_1 and β_2 *a priori* independent.
Model H_2:

$$\begin{aligned} y_i &= \gamma_1 + \gamma_2 x_i + \gamma_3 x_i^2 + \epsilon_i \\ &\epsilon_i \sim N(0,1) \\ \gamma_1 &= N(0,1), \quad \gamma_2 \sim N(1,1), \quad \gamma_3 \sim N(0,1), \end{aligned}$$

with $\gamma_1, \gamma_2, \gamma_3$ *a priori* indendendent.

5a*. [5] Find the marginal distribution $p(y|H_1) = \int p(y|\beta)p(\beta)d\beta$. and $p(y|H_2) = \int p(y|\gamma)p(\gamma)d\gamma$.

5b. [5] Write down the Bayes factor $B = p(y|H_2)/p(y|H_1)$ for comparing model H_1 vs. model H_2 and evaluate it for the given data set.

5c. [5] We now replace the prior distributions by improper constant priors: $p(\beta) = c_1$ in model H_1; and $p(\gamma) = c_2$ in model H_2. We can still form evaluate integrals $\int p(y|\beta)p(\beta)d\beta$ and $\int p(y|\gamma)p(\gamma)d\gamma$ [2] and define a Bayes factor

$$\tilde{B} = \frac{\int p(y|\gamma)p(\gamma)d\gamma}{\int p(y|\beta)p(\beta)d\beta}.$$

Show that the value of the Bayes factor $\tilde{B}$ depends on the – arbitrarily chosen – constants c_1 and c_2.

[2] Although the marginal distributions might be improper, i.e. meaningless.