Midterm 2 – Solutions

3[10] Marginal under H_1: $p(y|H_1) = \int \left[\prod_{i=1}^n Poi(y_i|\lambda N_i)\right] Ga(\lambda|1,\beta) d\lambda$.
Under H_2: $p(y|H_2) = \int \left[\prod_{i=1}^n Bin(y_i|N_i,\theta)\right] Be(\theta|1,\beta-1) d\theta$.
Let $N = \sum N_i$ and $S = \sum y_i$.

$$\begin{aligned} p(y|H_1) &= \frac{\prod N_i^{y_i}}{\prod y_i!} \int \lambda^S e^{-N\lambda} \beta e^{-\beta\lambda} d\lambda = \frac{\prod N_i^{y_i}}{\prod y_i!} \beta \int \lambda^{S+1-1} e^{-\lambda(N+\beta)} d\lambda \\ &= \frac{\prod N_i^{y_i}}{\prod y_i!} \beta \Gamma(S+1)(N+\beta)^{-(S+1)}. \end{aligned}$$

and

$$\begin{aligned} p(y|H_2) &= \frac{\prod N_i!}{\prod y_i!(N_i-y_i)!} \int \theta^S (1-\theta)^{N-S} (\beta-1)\theta^0 (1-\theta)^{\beta-1-1} d\theta \\ &= \frac{\prod N_i!}{\prod y_i!(N_i-y_i)!} \frac{\Gamma(S+1)\Gamma(N+\beta-S-1)}{\Gamma(N+\beta)} (\beta-1). \end{aligned}$$

Hence

$$B = \frac{(1-\beta)\prod\left[\frac{N_i!}{(N_i-y_i)!}\frac{\Gamma(N+\beta-S-1)}{\Gamma(N+\beta)}\right]}{\beta \prod N_i^{y_i} (N+\beta)^{-S-1}}.$$

4a [10] Let $\Sigma = VV'$ be a choleski decomposition of Σ and $L = V^{-1}$. By multiplying from the left with L we rewrite the likelihood as $\underbrace{Ly}_{z} \sim N(\underbrace{LX}_{W}, I)$.

Find the posterior $p(\beta|y) \propto p(\beta)p(y|\beta,\Sigma)$ as:

$$\begin{aligned} p(\beta|y,\Sigma) &\propto \exp\left(-\frac{1}{2}(\beta-\mu)'T^{-1}(\beta-\mu)\right) \exp\left(-\frac{1}{2}(z-W\beta)'(z-W\beta)\right) \\ &\propto e^{-\frac{1}{2}[\beta'T^{-1}\beta - 2\beta'T^{-1}\mu + \beta'(W'W)\beta - 2\beta'W'z]} \\ &\propto e^{-\frac{1}{2}[\beta' \overbrace{(T^{-1}+W'W)}^{S^{-1}} \beta - 2\beta' S^{-1} \overbrace{S(T^{-1}\mu + W'z)}^{m}]} \\ &\propto e^{-\frac{1}{2}(\beta-m)'S^{-1}(\beta-m)} = N(m,S). \end{aligned}$$

4b. [5] Note that $y|\beta \sim N(X\beta, \sigma^2 I)$ and $\beta \sim N(\mu, \sigma^2 R)$ imply (by Lemma 1 on the homework 6 solutions): $y \sim N(X\mu, \sigma^2 I + \sigma^2 XRX'))$, where I is a $n \times n$ identity matrix. Hence

$$\begin{aligned} p(\sigma^2|y) &\propto p(y|\sigma^2)p(\sigma^2) \\ &= N(y|X\mu, \sigma^2(I+XRX'))\text{Inv-}\chi^2(\sigma^2|\nu_0, s_0) \\ &\propto \left(\sigma^2\right)^{-\frac{n}{2}} \exp[\frac{1}{2}(y-X\mu)'(I+XRX')(y-X\mu)] \cdot (\sigma^2)^{-(\nu_0/2+1)} e^{-\frac{1}{2}\nu_0 s_0 \frac{1}{\sigma^2}} \\ &= \left(\sigma^2\right)^{-(\frac{\nu_0+n}{2}+1)} e^{-\frac{1}{2}\frac{1}{\sigma^2}\left(\nu_0 s_0 + \underbrace{(y-X\mu)'(I+XRX')(y-X\mu)}_{nS}\right)} \\ &= \text{Inv-}\chi^2\left(n+\nu_0, \frac{\nu_0 s_0 + nS}{n+\nu_0}\right). \end{aligned}$$

5a. [5] Denote with X_1 the $n \times 2$ design matrix with 1 in the first column and x in the second column, and with X_2 the $n \times 3$ design matrix with columns 1,x and x^2. Let $\mu_1 = (0,1)'$, $\mu_2 = (0,1,0)$.
Then (by Lemma 1 on the HW6 solution) $p(y|H_1) = N(y|X_1\mu_1, I + X_1X_1')$ and $p(y|H_2) = N(y|X_2\mu_2, I + X_2X_2')$.

5b. [5]

$$B = \frac{N(y|X_2\mu_2, I + X_2X_2')}{N(y|X_1\mu_1, I + X_1X_1')} = \frac{0.001907}{0.00579} = 0.33.$$

5c. [5]

$$\begin{aligned}
\text{“}p(y|H_1)\text{''} &= \int p(y|X_1\beta, I)p(\beta)d\beta \\
&= \int (2\pi)^{-\frac{n}{2}} \exp(\frac{1}{2}(y - X_1\beta)'(y - X_1\beta)) \\
&= c_1(2\pi)^{-\frac{n}{2}} \int \exp(-\frac{1}{2}[y'y - 2\beta' X_1' y + \beta' X_1' X_1 \beta] d\beta \\
&= c_1(2\pi)^{-\frac{n}{2}} \exp[-\frac{1}{2}y'y] \cdot \\
&\quad \int \exp[-\frac{1}{2}[\beta'(X_1'X_1)\beta - 2\beta(X_1'X_1)\underbrace{(X_1'X_1)^{-1}X_1'y}_{\hat\beta} + \hat\beta'(X_1'X_1)\hat\beta]d\beta \exp[\frac{1}{2}\hat\beta'(X_1'X_1)\hat\beta] \\
&= c_1(2\pi)^{-\frac{n}{2}} \exp[-\frac{1}{2}y'y + \frac{1}{2}\hat\beta'(X_1'X_1)\hat\beta]|(X_1'X_1)|^{-\frac{1}{2}}.
\end{aligned}$$

Alternatively use the “candidate formula” (p129, equation 5.5) with $\beta = \hat\beta$.
And by the same argument

$$\int p(y|X_2\gamma, I)p(\gamma)d\gamma = c_2(2\pi)^{-\frac{n}{2}} \exp[-\frac{1}{2}y'y + \frac{1}{2}\hat\gamma'(X_2'X_2)\hat\gamma]|(X_2'X_2)|^{-\frac{1}{2}}$$

and hence

$$B = \frac{c_1 \exp[\frac{1}{2}\hat\beta'(X_1'X_1)\hat\beta]|(X_1'X_1)|^{-\frac{1}{2}}}{c_2 \exp[\frac{1}{2}\hat\gamma'(X_2'X_2)\hat\gamma]|(X_2'X_2)|^{-\frac{1}{2}}}.$$