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RANDOM MEASURES WITH AFTEREFFECTS

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A class of $\mathscr{D}$ of random measures, generalizing the class of completely random measures, is developed and shown to contain the class of Poisson cluster point processes. An integral representation is obtained for $\mathscr{D}$, generalizing the Lévy-Itô representation for processes with independent increments. A subclass $\mathscr{D}_n \subset \mathscr{D}$ is defined such that for $X \in \mathscr{D}_n$, the distribution of the random vector $X(A_1), \cdots, X(A_m)$, $m > n$, $A_1, \cdots, A_m$ disjoint, is determined by the distributions of all subvectors $X(A_{i_1}), \cdots, X(A_{i_k})$, $1 \leqq k \leqq n$. The class $\mathscr{D}_1$ coincides with the class of completely random measures.

1. Introduction and summary. The purpose of this article is to provide a general probabilistic framework for infinitely divisible (i.d.) random measures X which are not *completely random* (Definition 2.9). Since a completely random (c.r.) measure takes on values independently on disjoint sets, its distribution is fully specified by its one-dimensional distributions. The presence of dependence, however, compels that the finite dimensional distributions of all vectors $\mathbf{X}(\mathbf{A}_m) = (X(A_1), \cdots, X(A_m))$, $m \geqq 1$, be specified to determine the distribution of the measure. It is shown that, in the particular case of limited aftereffects called *n-dependence* (defined in Section 2), only the finite dimensional distributions of dimensions $1, \cdots, n$ are necessary to yield the probability law of the random measure (Theorem 3.10). Aside from possibly a deterministic component, these random measures are purely atomic, and an integral representation (Theorem 3.7) generalizing the Lévy–Itô representation for processes with independent increments is also presented.

The study of random measures is motivated by the study of the class of nonnegative, nondecreasing stochastic processes $Y(t)$ defined on a real parameter t, where $X(s, t] = Y(t) - Y(s)$, $s < t$, defines the unique Stieltjes measure X corresponding to Y. The property of Y having independent increments is equivalent to X being c.r. When $Y(t)$ does not have independent increments X is said to be subject to aftereffects, and hereafter the class of random measures which are not necessarily c.r. will be called *random measures with aftereffects*.

Kingman (1967), in his study of c.r. measures, provides a canonical representation for the moment generating function (m.g.f.) of $X(A)$ based on the Lévy representation for i.d. random variables. He also shows that under a weak

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finiteness assumption X is the sum of a deterministic component and a purely atomic, i.d. component.

When X has aftereffects, however, the interdependence among $X(A_1), \cdots, X(A_m)$, $m \geqq 2$, leads here to the use of the probability generating functional as an analytic tool. The elegant representation of this functional for nonnegative, i.d. stochastic processes given by Lee (1967) is utilized for the study of random measures with aftereffects.

The class of random measures constructed in Section 3 is shown to contain several important classes of stochastic processes, including the set of all c.r. measures, hence all nonnegative, i.d. processes with independent increments. In addition, it is shown to contain the family of Poisson cluster point processes, including the Neyman–Scott, Bartlett–Lewis, Vere–Jones, and Gauss–Poisson processes. See, for example, Daley and Vere–Jones (1972).

2. The probability generating functional of a random measure. The ideas developed in this and the following sections draw upon several important papers on the theory of stochastic processes. The consistency conditions are essentially those used by Ferguson (1973) in his treatment of Dirichlet processes. The preliminary results on generating functionals are based upon papers by Westcott (1972) and Jagers (1972) on the theory of stochastic point processes. Lee's general representation for the multivariate m.g.f. of a nonnegative i.d. stochastic process is central to the development in Section 3. Finally, the functional given by (3.1) is a generalization of the probability generating functional of a regular i.d. stochastic point process, as given in Ammann and Thall (1977).

Some preliminary notation is required. Let T be a σ-compact metric space with σ-field $\mathscr{T}$ of subsets of T, and take $(\Omega, \mathscr{F}, P)$ to be any probability space. Write $R_+ = [0, \infty)$ and $S_+ = [0, \infty]$, the one point compactification of R_+ by ∞, with corresponding Borel σ-fields $\mathscr{R}_+ = \mathscr{B}(R_+)$ and $\mathscr{S}_+ = \mathscr{B}(S_+)$. The set of all nontrivial measures μ defined on $\mathscr{T}$ and finite on bounded sets is denoted by $\mathscr{M}$, with $\mathscr{A}$ the smallest σ-ring generated by the Borel cylinder sets $\{\mu : \mu(A_i) \in B_i, 1 \leqq i \leqq m\}$, $A_i \in \mathscr{T}$, $B_i \in \mathscr{S}_+$, $1 \leqq i \leqq m$. A *random measure* X w.r.t. the above probability space is defined to be any $\mathscr{F}$-measurable function X: $\Omega \rightarrow \mathscr{M}$. By a standard abuse of notation, $X(\omega, A)$ will be taken to mean $\mu(A)$ for $\mu = X(\omega)$, $\omega \in \Omega$, with the argument ω suppressed in most instances. Since by assumption $X(\omega, A) < \infty$ for all bounded $A \in \mathscr{T}$ and all $\omega \in \Omega$, it follows from the σ-compactness of T that each $X(\omega)$ is σ-finite.

The finite dimensional distributions of X must satisfy certain consistency conditions to insure that they define a unique probability law $\mathscr{P}$ on $(\mathscr{M}, \mathscr{A})$. The conditions given below are those of Ferguson ((1973), page 213) with the addition of a further condition to insure that X is countably additive.

Consistency conditions.

(i) The distribution of $X(A_1), \cdots, X(A_m)$ is identical to the distribution of

$X(A_{i_1}), \cdots, X(A_{i_m})$ for all $m \geqq 1$, $A_1, \cdots, A_m \in \mathscr{T}$ and permutations $(i_1, \cdots, i_m)$ of $(1, \cdots, m)$.

(ii) For arbitrary $A_1, \cdots, A_r \in \mathscr{T}$, form the $k = 2^r$ sets obtained by taking all intersections of the A_j and their complements; i.e., define $B_{\mathbf{i}} = \bigcap_1^r A_j^{i_j}$, where $\mathbf{i} = (i_1, \cdots, i_r)$, $i_j = 0$ or 1, and $A_j^1 = A_j$, $A_j^0 = T \backslash A_j$. Note that $\{B_{\mathbf{i}}\}$ forms a partition of T. The distribution of $\mathbf{X}(\mathbf{A}_r)$ is then obtained from the distribution of $\{X\{B_{\mathbf{i}})\}$ by defining

$$X(A_j) = \sum_{\{\mathbf{i}: i_j = 1\}} X(B_{\mathbf{i}}) .$$

(iii) For arbitrary disjoint $A_1, \cdots, A_m \in \mathscr{T}$, if $B_1 = \bigcup_1^{r_1} A_i, \cdots, B_k = \bigcup_{r_{k-1}+1}^{r_k} A_i$, $1 \leqq r_1 < \cdots < r_k = m$, then the distribution of $\sum_1^{r_1} X(A_i), \cdots, \sum_{r_{k-1}+1}^{r_k} X(A_i)$ is identical to the distribution of $X(B_1), \cdots, X(B_k)$.

(iv) If $\{A_j\}$ is a monotone decreasing sequence from $\mathscr{T}$ such that $A_j \downarrow \phi$ and $X(A_1) < \infty$ a.s., then $X(A_j) \downarrow 0$ a.s.

Note that a.s. σ-additivity for X follows from (iv), while (ii) implies $X(\phi)$ is degenerate at 0.

LEMMA 2.1 (Ferguson (1973), page 213). *If the finite dimensional distributions of X satisfy conditions* (i)—(iv) *then there exists a unique probability measure $\mathscr{P}$ on $(\mathscr{M}, \mathscr{A})$ yielding these distributions.*

With these preliminaries established, the probability generating functional of a random measure is now defined. Intuitively, the probability generating functional of a random measure describes its probabilistic structure in a manner analogous to that in which a m.g.f. describes the law of a random variable.

DEFINITION 2.2. The *probability generating functional* of a random measure X is

$$G[\xi] = Ee^{X \circ \log \xi} , \tag{2.1}$$

where $\mu \circ f = \int_T f \, d\mu$ and the expectation is taken w.r.t. P. If $\xi \equiv 0$ on some set A then $\exp(X \circ \log \xi)$ is taken to be 0, unless $X(A) = 0$ (a.s.) in which case it is taken to be $\exp[X \circ (I_{T \backslash A} \log \xi)]$.

Two classes of functions for which (2.1) is meaningful are given by the following definition.

DEFINITION 2.3. (i) V is the class of all $\mathscr{T}$-measurable functions $\xi : T \to [0, 1]$ such that $1 - \xi$ vanishes outside a compact set and (ii) V_0 is the subclass of functions in V which are bounded away from 0.

V_0 contains virtually all functions which are of interest here in the sense that, while $G[\xi]$ exists for all $\xi \in V$, it is nontrivial for all $\xi \in V_0$. Define for positive integer m, disjoint sets $A_1, \cdots, A_m \in \mathscr{T}$, and $u_1, \cdots, u_m \in R_+$ the special element of V_0

$$\eta_m = 1 - \sum_{i=1}^m (1 - e^{-u_i}) I_{A_i} , \tag{2.2}$$

where $I_A(t) = 1$ if $t \in A$, 0 otherwise. In all that follows statements involving

η_m shall be understood to be given for arbitrary m, $A_1, \cdots, A_m$, and $u_1, \cdots, u_m$ as given above. Note that $G[\eta_m]$ is the m.g.f. of $\mathbf{X}(\mathbf{A}_m)$.

LEMMA 2.4. *G is uniquely determined on V_0 by $\mathscr{S}$, and conversely.* (See Westcott (1972).)

The next two theorems give conditions on G for determination of finite dimensional distributions and weak convergence of the corresponding random measures. The following continuity lemma for G is first required.

LEMMA 2.5. *If G is a probability generating functional and $\{\xi_n\}$ is a sequence in V_0 satisfying the conditions*

(i) *$\{1 - \xi_n\}$ vanish outside a common bounded set,*
(ii) *$\{\xi_n\}$ is uniformly bounded away from 0, and*
(iii) *$\xi_n \to \xi$ pointwise,*

then $G[\xi_n] \to G[\xi]$.

If $X(A_m) = \infty$ a.s. for some (necessarily) unbounded $A_m \in \mathscr{T}$, then finding the joint distribution of $\mathbf{X}(\mathbf{A}_m)$ effectively reduces to finding the joint distribution of $X(A_1), \cdots, X(A_{m-1})$. The determination of finite dimensional distributions of X through evaluation of G at η_m, when some A_i's are unbounded, may thus be carried out w.l.o.g. under the assumption that $P[X(A_i) < \infty] > 0$, or equivalently $G[1 - zI_{A_i}] > 0$, $0 < z < 1$, $1 \leqq i \leqq m$.

THEOREM 2.6. *A functional G on V is the probability generating functional of a random measure* iff, *for each η_m, $G[\eta_m]$ is the* m.g.f. *of a random m-vector of non-negative components.*

PROOF. Necessity is immediate from the definition of G. Sufficiency follows from verification of the consistency conditions.

Condition (i) is obvious. To show (ii), take arbitrary $A_1, \cdots, A_r$ and $\{B_{\mathbf{i}}\}$ as defined in the condition, and let

$$\xi = 1 - \sum_{\{\mathbf{i} \neq \mathbf{0}\}} [1 - \exp(-u_{\mathbf{i}})] I_{B_{\mathbf{i}}} .$$

The joint distribution of the $B_{\mathbf{i}}$'s is then determined by $G[\xi]$, and it follows that $G[\xi]$ is the m.g.f. of $\mathbf{X}(\mathbf{A}_r)$ by setting $u_{\mathbf{i}} = u_j$ for all $\mathbf{i}$ such that $i_j = 1$ and defining

$$X(A_j) = \sum_{\{\mathbf{i}: i_j = 1\}} X(B_{\mathbf{i}}) , \qquad 1 \leqq j \leqq r .$$

To show (iii), take $\mathbf{A}_m$ and $\mathbf{B}_k$ as defined in the condition and note that, for $u_{r_{j-1}+1} = \cdots = u_{r_j} = v_j$, $1 \leqq j \leqq k$,

$$1 - \sum_{j=1}^{k} (1 - e^{-v_j}) I_{B_j} = \eta_m .$$

Finally, let $\{A_j\}$ be a sequence of sets as in (iv), so that $G[\xi_j] > 0$ for $\xi_j = 1 - zI_{A_j}$ and fixed z, $0 < z < 1$. Since $\xi_j \to 1$ pointwise, it follows from Lemma 2.5 that $G[\xi_j] \to G[1] = 1$. Hence $X(A_j) \downarrow 0$ in law, and so a.s.

The finite dimensional distributions generated by G thus extend uniquely to define a probability law $\mathscr{P}$ for a random measure X. Since the probability generating functional G^* of X must agree with G for all simple functions $\xi \in V_0$, it follows that $G^*[\xi] = G[\xi]$ for all $\xi \in V_0$. □

The probability generating functional of a random measure X thus embodies the law $\mathscr{P}$ of X through the functions $\eta_m \in V_0$. The analogous result for stochastic point processes is given by Westcott ((1972), Theorem 4).

A sequence of random measures $\{X_n\}$ with respective probability measures $\{\mathscr{P}_n\}$ is said to *converge weakly* to a random measure X with probability measure $\mathscr{P}$ iff $\mathscr{P}_n \circ h \rightarrow \mathscr{P} \circ h$ for all bounded continuous functions h on T. Denote weak convergence $X_n \rightarrow X(w)$. The following result allows weak convergence to be expressed in terms of probability generating functionals. Convergence of the sequence of probability generating functionals $\{G_n\}$ is taken here to mean that $G_n[\xi]$ converges for all $\xi \in V_0$.

THEOREM 2.7. *If $X_n \rightarrow X(w)$, then the corresponding probability generating functionals G_n converge to the probability generating functional G of X. Conversely, if a sequence of probability generating functionals $\{G_n\}$ of random measures $\{X_n\}$ converges to a functional G continuous for all sequences $\xi_n \rightarrow 1$ which satisfy the conditions of Lemma 2.3, then G is the probability generating functional of a random measure X and $X_n \rightarrow X(w)$.*

PROOF. This result is due to Jagers ((1972), Theorem 1), stated there in terms of characteristic functionals. □

The remainder of this paper is concerned with investigation of purely atomic i.d. random measures with aftereffects, and the analysis is carried out exclusively via probability generating functionals. Lee (1967) has given a characterization of nonnegative i.d. stochastic processes in terms of their finite dimensional distributions, although his representation is expressed here equivalently in terms of probability generating functionals. The result is stated for random measures.

THEOREM 2.8 (Lee (1967), page 150). *A random measure is* i.d. iff *its probability generating functional takes the form*

$$\log G[\xi] = \alpha \circ \log \xi + \int_{\mathscr{M}} [e^{\mu \circ \log \xi} - 1]\tilde{P}(d\mu)\,, \tag{2.3}$$

where α is a measure on $(T, \mathscr{T})$ and $\tilde{P}$ is a unique measure on $(\mathscr{M}, \mathscr{A})$, such that $G[\eta_m] > 0$.

By convention, $\tilde{P}$ is called the KLM (Kerstan–Lee–Matthes) measure of the process. A random measure for which $\alpha \equiv 0$ in (2.3) is said to be centered. Note that any i.d. random measure can be centered, and that $\tilde{P}$ determines X up to the additive deterministic component given by α. Since the present article is concerned with the stochastic structure of X, and hence with $\tilde{P}$, it is hereafter assumed that all random measures under consideration are centered.

Denote by $\mu_{\mathbf{t}_k,\mathbf{v}_k}$ the element of $\mathscr{M}$ which gives mass $v_j > 0$ to the point $t_j \in T$ for distinct $t_1, \cdots, t_k$. That is,

$$\mu_{\mathbf{t}_k,\mathbf{v}_k} = \sum_{j=1}^k v_j \delta_{t_j} ,$$

where $\delta_t(A) = 1$ if $t \in A$, 0 otherwise.

DEFINITION 2.9. A random measure X is said to be *completely random* iff for each collection $A_1, \cdots, A_m$ of disjoint sets in $\mathscr{T}$, $X(A_1), \cdots, X(A_m)$ are independent, $m \geqq 2$.

Theorem 2.10 shows that a random measure is c.r. precisely when its KLM measure is concentrated on the set of measures with a single atom.

THEOREM 2.10. *A random measure is* c.r. iff *its probability generating functional takes the form* (2.3) *with* $\tilde{P}$ *concentrated on* $D_1 = \{\mu = \mu_{t,v} : t \in T, v > 0\}$.

PROOF. Suppose $\tilde{P}$ is concentrated on D_1. Since $\tilde{P}$ is a measure, the function

$$Q(A, B) = \tilde{P}[\mu_{t,v} : t \in A, v \in B] ,$$

defined for $A \in \mathscr{T}$ and $B \in \mathscr{R}_+$, defines a measure $Q(A \times B) = Q(A, B)$ on $(T \times R_+, \mathscr{T} \times \mathscr{R}_+)$. It follows that

$$\int_{\mathscr{M}} [e^{\mu \circ \log \eta_m} - 1]\tilde{P}(du) = \sum_{j=1}^m \int_{R_+} [\exp(-u_j v) - 1]Q(A_j, dv) \tag{2.4}$$

which implies that $X(A_1), \cdots, X(A_m)$ are independent.

Conversely, if X is c.r., then it is i.d. (Kingman (1967), page 64). Thus for each $A \in \mathscr{T}$, there exists a measure $Q_A^*(\cdot) = Q^*(A, \cdot)$ on $(R_+, \mathscr{R}_+)$ such that

$$-\log Ee^{-uX(A)} = \int_{R_+} [1 - e^{-uv}]Q^*(A, dv) .$$

(For details, see Lee (1967), page 149). Since $X(A_1), \cdots, X(A_m)$ are independent for $A_1, \cdots, A_m$ disjoint,

$$-\log G[\eta_m] = \sum_{j=1}^m \int_{R_+} [1 - e^{-u_j v}]Q^*(A_j, dv) .$$

Define the measure $\tilde{P}^*[\mu_{t,v} : t \in A, v \in B] = Q^*(A, B)$ on D_1. Equality (2.4) must then hold for $G[\eta_m]$ with Q and $\tilde{P}$ replaced by Q^* and $\tilde{P}^*$, respectively. By the uniqueness of $\tilde{P}$, however, $\tilde{P} = \tilde{P}^*$, so that $\tilde{P}$ must be concentrated on D_1. $\square$

It follows that any c.r. measure has probability generating functional of the form

$$\log G[\xi] = \int_{T \times R_+} [\xi^v(t) - 1]Q(dv, dt) , \tag{2.5}$$

where Q is defined as in the proof of Theorem 2.10. In the case of $T = R$ (real space), (2.5) is the Lévy representation of a stochastic process with independent increments.

The class of stochastic point processes is an important subclass of random measures, and the examples given below are provided as a foundation upon which is constructed the probability generating functional of a class of random measures with aftereffects. Formally, define $\mathscr{N} \subset \mathscr{M}$ to be the set of all non-negative, integer-valued measures N, and define the σ-ring $\mathscr{A}_{\mathscr{N}} = \mathscr{A} \cap \mathscr{N}$.

A stochastic point process X is then defined to be any random measure having all realizations in $\mathscr{N}$, such that $X(A) < \infty$ a.s. for all compact $A \in \mathscr{T}$.

The following result is immediate from expression (2.5) and the definition of $\mathscr{N}$.

COROLLARY 2.11. *The probability generating functional of a* c.r. *stochastic point process takes the form*

$$\log G[\xi] = \textstyle\sum_{k=1}^{\infty} \int_T [\xi^k(t) - 1]Q_k(dt)\,, \tag{2.6}$$

where each Q_k is a measure on $(T, \mathscr{T})$ which is finite on compact sets.

If N_k is a Poisson process having intensity measure Q_k, then the process given by (2.6) is the superposition

$$X = \textstyle\sum_{k=1}^{\infty} kN_k\,.$$

That is, X is the limit of finite superpositions of (independent) Poisson processes, the kth of which has atoms all of mass k.

The Gauss–Poisson (GP) process, first introduced by Newman (1970), has probability generating functional

$$\begin{aligned} \log G[\xi] &= \textstyle\int_T [\xi(t) - 1]H_1(dt) + \frac{1}{2} \int_{T^2} [\xi(t) - 1][\xi(s) - 1]H_2(dt \times ds) \\ &= \textstyle\int_T [\xi(t) - 1]\Lambda_1(dt) + \frac{1}{2} \int_{T^2} [\xi(t)\xi(s) - 1]\Lambda_2(dt \times ds)\,, \end{aligned} \tag{2.7}$$

where $\Lambda_1(dt) = H_1(dt) - H_2(dt \times T)$ and $\Lambda_2 = H_2$. Milne and Westcott (1972) state that a GP process is c.r. iff H_2 is concentrated on the diagonal set $\{(t, s): t = s\}$ of T^2. In this case, (2.7) has the form given by (2.6) with $Q_1 = \Lambda_1$, $Q_2 = \Lambda_2$, and $Q_k \equiv 0$ for all $k > 2$.

Denote by D_n the set $\{\mu_{\mathbf{t}_k, \mathbf{v}_k} : 1 \leqq k \leqq n\}$ of all nontrivial measures having n or fewer atoms. Take $\mathscr{D}_n$ and $\mathscr{D}$ to be the classes of random measures with KLM measure concentrated on D_n and $D = \bigcup_1^{\infty} D_r$, respectively. Theorem 2.9 shows that $\mathscr{D}_1$ is the class of c.r. measures, and it can be seen from (2.7) that the GP process is an element of $\mathscr{D}_2$. Ammann and Thall (1977) have given conditions on $\{Q_k\}$ under which (2.8) is the probability generating functional of a regular i.d. stochastic point process:

$$\log G[\xi] = \textstyle\sum_{k=1}^{\infty} \frac{1}{k!} \int_{T^k} \prod_{i=1}^{k} [\xi(t_i) - 1]Q_k(d\mathbf{t}_k)\,, \tag{2.8}$$

where $d\mathbf{t}_k = dt_1 \times \cdots \times dt_k$. The class of regular i.d. stochastic point processes (i.e., those for which $\tilde{P}[X(R) = \infty] = 0$) is the class of Poisson cluster processes. (See, for example, Kerstan, Matthes, and Mecke (1974).) The class $\mathscr{D}_n$ will hereafter be referred to as the class of *n-dependent random measures*, for reasons which shall be motivated by Theorem 3.10.

Notice that (2.5) generalizes a Poisson process to a c.r. measure by allowing atoms of all (positive) magnitudes, while (2.8) generalizes a Poisson process by allowing aftereffects in the form of clusters. It is shown in Section 3 that each element of $\mathscr{D}_n$ has a probability generating functional which embodies both of these generalizations.

3. Random measures with aftereffects. Let $\{Q_k\}$ be a sequence of Borel signed measures defined, respectively, on $(T^k \times R_+{}^k, \mathscr{T}^k \times \mathscr{R}_+{}^k)$. Consider the functional

$$(3.1) \qquad \log G_\infty[\xi] = \sum_{k=1}^\infty \frac{1}{k!} \int_{R_+^k \times T^k} \prod_{i=1}^k [\xi^{v_i}(t_i) - 1] Q_k(d\mathbf{t}_k; d\mathbf{v}_k),$$

defined on V. Theorem 3.2 gives necessary and sufficient conditions on the Q_k's for this functional to be the probability generating functional of an i.d. random measure. A preliminary lemma is required.

LEMMA 3.1. $\log G_\infty[\xi] > -\infty$ *for all* $\xi \in V_0$ iff

$$\sum_{k=1}^\infty \frac{1}{k!} \int_{R_+^k} \prod_{i=1}^k [e^{-uv_i} - 1] Q_k(A^k; d\mathbf{v}_k) > -\infty$$

for all $u > 0$ *and compact* $A \in \mathscr{T}$.

PROOF. Necessity follows by taking $\xi = 1 - (1 - e^{-u}) I_A$. For sufficiency, take compact $A \in \mathscr{T}$ and $\xi \in V_0$ such that $1 - \xi$ vanishes outside of A. If $u = -\log(\inf_{t \in A} \xi(t))$, then $u < \infty$ and $\xi^v(t) - 1 \geqq e^{-uv} - 1$ for all $v > 0$, which implies the desired result. □

It is assumed w.l.o.g. in what follows that each Q_k is symmetric, in the sense that $Q_k(A_1, \cdots, A_k; B_1, \cdots, B_k) = Q_k(A_{i_1}, \cdots, A_{i_k}; B_{i_1}, \cdots, B_{i_k})$ for each permutation $(i_1, \cdots, i_k)$ of $(1, \cdots, k)$, and also that Q_k places no mass on the set $\{\mathbf{t}_k : t_i = t_j \text{ for some } i \neq j\} \times R_+{}^k$.

THEOREM 3.2. G_∞ *is the probability generating functional of an* i.d. *random measure* iff

$$\text{(i)} \qquad 0 \leqq \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k!} \int_{R_+^k} \prod_{i=1}^k [1 - e^{-uv_i}] Q_k(A^k; d\mathbf{v}_k) < \infty$$

for all $u > 0$ *and compact* $A \in \mathscr{T}$, *and*

$$\text{(ii)} \qquad 0 \leqq \Lambda_k(B, C) =_{\text{def}} \sum_{m=k}^\infty \frac{(-1)^{m-k}}{(m-k)!} Q_m(B \times T^{m-k}; C \times R_+{}^{m-k})$$

for all $B \in \mathscr{T}^k$, $C \in \mathscr{R}_+{}^k$ *and* $k \geqq 1$.

PROOF. The expression given in (i) is $-\log Ee^{-u \cdot X(A)}$, which is finite since $X(A) < \infty$, a priori.

The easily verified equality

$$\prod_{i=1}^k (a_i - 1) = \sum_{m=1}^k (-1)^{k-m} \sum_{1 \leq i_1 < \cdots < i_m \leq k} \left[\prod_{j=1}^m a_{i_j} - 1\right]$$

and the symmetry of the Q_k's imply that

$$(3.2) \qquad \log G_\infty[\xi] = \sum_{k=1}^\infty \frac{1}{k!} \int_{R_+^k \times T^k} \sum_{m=1}^k \binom{k}{m} (-1)^{k-m} \left[\prod_{i=1}^m \xi^{v_i}(t_i) - 1\right] Q_k(d\mathbf{t}_k; d\mathbf{v}_k)$$

$$= \sum_{m=1}^\infty \frac{1}{m!} \int_{R_+^m \times T^m} \left[\prod_{i=1}^m \xi^{v_i}(t_i) - 1\right] \Lambda_m(d\mathbf{t}_m; d\mathbf{v}_m).$$

Since X is i.d., it has a unique KLM measure $\tilde{P}$ satisfying

$$\log G_\infty[\xi] = \int_{\mathscr{M}} [e^{\mu \circ \log \xi} - 1]\tilde{P}(d\mu) . \tag{3.3}$$

Furthermore, since $e^{\mu \circ \log \xi} = \prod_1^k \xi^{v_i}(t_i)$ iff $\mu = \mu_{\mathbf{t}_k, \mathbf{v}_k}$, it follows that $\tilde{P}$ must be concentrated on $D = \bigcup_1^\infty D_n = \bigcup_1^\infty J_n$, where $J_n = D_n \backslash D_{n-1}$ $(D_0 = \varnothing)$. For each $k \geqq 1$, define the measure

$$\Lambda_k^*(\mathbf{A}_k; \mathbf{B}_k) = k!\, \tilde{P}[\mu \in J_k : \mathbf{t}_k \in \mathbf{A}_k, \mathbf{v}_k \in \mathbf{B}_k] \tag{3.4}$$

on $(T^k \times R_+{}^k, \mathscr{T}^k \times \mathscr{R}_+{}^k)$, assumed w.l.o.g. to be symmetric in the same sense as Q_k. Expression (3.3) may thus be written in the form of (3.2), with Λ_k^* in place of Λ_k for each $k \geqq 1$. It follows from the uniqueness of the representation that $\Lambda_k \equiv \Lambda_k^*$, for each k, whence condition (ii) follows.

To show sufficiency, note that condition (i) and Lemma 3.1 together imply that $\log G_\infty[\xi] > -\infty$ for all $\xi \in V_0$. If $\{\xi_n\}$ and ξ satisfy the conditions of Lemma 2.5, then the argument given in the proof of Lemma 3.1 and the dominated convergence theorem imply $G_\infty[\xi_n] \to G_\infty[\xi]$. For compact $A_1, \cdots, A_m \in \mathscr{T}$,

$$\begin{aligned} \log G_\infty[\eta_m] = \textstyle\sum_{k=1}^\infty \frac{1}{k!} \sum_{r=1}^k \sum_{a_1+\cdots+a_m=r} \binom{r}{\mathbf{a}_m} \int_{R_+^r} [\exp(-\sum_{i=1}^m u_i w_i) - 1] \\ \times \Lambda_k(A_1^{a_1} \times \cdots \times A_m^{a_m} \times A^{k-r}; d\mathbf{v}_r \times R_+{}^{k-r}) , \end{aligned} \tag{3.5}$$

where $A = T \backslash (\bigcup_1^m A_i)$, $\binom{r}{\mathbf{a}_m} = r!/(a_1! \cdots a_m!)$, and $w_i = v_{a_{i-1}+1} + \cdots + v_{a_i}$. It follows from (ii) that (3.5) is the m.g.f. of an m-dimensional random vector with nonnegative components. By Theorem 2.6, G_∞ is the probability generating functional of a random measure. Under condition (ii), the equality (3.4), with Λ_k rather than Λ_k^*, defines a measure $\tilde{P}$ satisfying (3.3). By the uniqueness of the Lee representation, the process is i.d. □

The following corollary gives sufficient conditions for G_∞ to be a probability generating functional. Although stronger than the necessary and sufficient condition of Theorem 3.2, they are simpler and more easily verified.

COROLLARY 3.3. *If* $\{Q_k\}$ *is a sequence of measures, defined respectively on* $(T^k \times R_+{}^k, \mathscr{T}^k \times \mathscr{R}_+{}^k)$, *satisfying*

$$\text{(i)} \qquad \int_{R_+} [1 - e^{-uv}] Q_1(A, dv) < \infty$$

for all $u > 0$ *and compact* $A \in \mathscr{T}$, *and*

$$\text{(ii)} \qquad Q_k(\mathbf{A}_k; \mathbf{B}_k) \geqq Q_{k+1}(\mathbf{A}_k \times T; \mathbf{B}_k \times R_+)$$

for all $\mathbf{A}_k \in \mathscr{T}_k$ *and* $\mathbf{B}_k \in \mathscr{R}_+{}^k$, $k \geqq 1$, *then* G_∞ *is the probability generating functional of an* i.d. *random measure.*

PROOF. Condition (ii) implies that

$$\sum_{k=m}^\infty \frac{(-1)^{k-m}}{(k-m)!} Q_k(\mathbf{A}_m \times T^{k-m}; \mathbf{B}_m \times R_+{}^{k-m}) \geqq 0$$

for all $\mathbf{A}_m \in \mathscr{T}^m$, $\mathbf{B}_m \in \mathscr{R}_+{}^m$. Conditions (i) and (ii) together imply that

$$\begin{aligned}
\left|\sum_{k=1}^{\infty} \frac{1}{k!} \int_{R_+^k} \prod_{i=1}^{k} (e^{-uv_i} - 1) Q_k(A^k; d\mathbf{v}_k)\right| &\leqq \sum_{k=1}^{\infty} \frac{1}{k!} \int_{R_+} (1 - e^{-uv}) Q_k(A^k; dv \times R_+{}^{k-1}) \\
&\leqq (e - 1) \int_{R_+} (1 - e^{-uv}) Q_1(A, dv) \\
&< \infty . \qquad \square
\end{aligned}$$

It is shown in the proof of Theorem 3.2 that the KLM measure of an i.d. random measure with probability generating functional G_∞ is concentrated on D, with Λ_k the measure defined by the relativization of $\tilde{P}$ to J_k. The random measure X with this probability generating functional may be expressed as the limit of superpositions of measures of the form $\mu_{\mathbf{t}_k, \mathbf{v}_k}$, chosen according to $\tilde{P}$. This interpretation is formalized by the following construction.

Let X be a random measure with probability generating functional G_∞ and let $\{T_m\}$ be a sequence of compact sets in $\mathscr{T}$ such that $T_m \uparrow T$. Define, for arbitrary $\varepsilon > 0$ and fixed $m \geqq 1$,

$$L_{m,\varepsilon} = \sum_{k=1}^{\infty} \frac{1}{k!} \Lambda_k(T_m{}^k; [\varepsilon, \infty)^k) .$$

Condition (i) of Theorem 3.2 implies

$$\begin{aligned}
L_{m,\varepsilon} &\leqq \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \Lambda_k(T_m \times T^{k-1}; [\varepsilon, \infty) \times R_+{}^{k-1}) \\
&= Q_1(T_m; [\varepsilon, \infty)) \\
&< \infty .
\end{aligned}$$

For each m and ε, take $N_{m,\varepsilon}$ to be a Poisson random variable with mean $L_{m,\varepsilon}$ and let $\{\mu_j\}$ be a sequence of independent random measures chosen independently of $N_{m,\varepsilon}$ with probabilities

$$P_{m,\varepsilon}[\mu_j = \mu_{\mathbf{t}_k, \mathbf{v}_k} : \mathbf{t}_k \in A, \mathbf{v}_k \in B] = \frac{\Lambda_k(A; B)}{k!\, L_{m,\varepsilon}} \tag{3.6}$$

for $A \subset T_m{}^k$, $B \subset [\varepsilon, \infty)^k$ and $k \geqq 1$. Define

$$X_{m,\varepsilon} = \sum_{j=0}^{N_{m,\varepsilon}} \mu_j$$

where $\mu_0 = 0$ a.s. Theorem 3.4 shows that the random measure having probability generating functional G_∞ is the limit of Poisson sums of the above form.

THEOREM 3.4. *If X is a random measure with probability generating functional G_∞, then $X_{m,\varepsilon} \to X(w)$ as $m \to \infty$ and $\varepsilon \downarrow 0$.*

PROOF. For fixed m and ε,

$$\begin{aligned}
Ee^{\mu \circ \log \xi} &= L_{m,\varepsilon}^{-1} \sum_{k=1}^{\infty} \int_{[\varepsilon,\infty)^k} \int_{T_m^k} \prod_{i=1}^{k} \xi^{v_i}(t_i) \Lambda_k(d\mathbf{t}_k; d\mathbf{v}_k) \\
&=_{\text{def}} M_{m,\varepsilon}[\xi] .
\end{aligned}$$

It follows that

$$\begin{aligned} G_{m,\varepsilon}[\xi] &= E\{\exp[X_m \circ \log \xi]\} \\ &= E\{(M_{m,\varepsilon}[\xi])^{N_{m,\varepsilon}}\} \\ &= \sum_{k=0}^{\infty} \frac{(M_{m,\varepsilon}[\xi])^k}{k!} L_{m,\varepsilon}^k \exp[-L_{m,\varepsilon}] \\ &= \exp\left[\sum_{k=1}^{\infty} \frac{1}{k!} \int_{[\varepsilon,\infty)^k} \int_{T_m^k} \left(\prod_{i=1}^k \xi^{v_i}(t_i) - 1\right)\Lambda_k(d\mathbf{t}_k; d\mathbf{v}_k)\right]. \end{aligned}$$

Since $G_{m,\varepsilon} \to G_\infty$, Theorem 2.7 implies that $X_{m,\varepsilon} \to X(w)$. □

The probability law of any random measure having probability generating functional G_∞ may thus be viewed as the limit of Poisson convolutions of laws given by (3.6). This idea was first introduced by Kerstan and Matthes (1964) for stationary i.d. stochastic point processes.

For arbitrary $A \in \mathscr{T}$ and $\delta > 0$, define $Z(A, \delta)$ to be the number of atoms of X in A with mass $\geqq \delta$, and define $Z_{m,\varepsilon}$ analogously for $X_{m,\varepsilon}$. It follows that $Z_{m,\varepsilon} \to Z(w)$, where Z is a random measure on $(T \times R_+, \mathscr{T} \times \mathscr{R}_+)$ with probability generating functional given by the following theorem.

THEOREM 3.5. *If X has probability generating functional G_∞, then Z has probability generating functional*

$$\log G[\xi] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{R_+^k \times T^k} \prod_{i=1}^k [\xi(t_i, v_i) - 1] Q_k(d\mathbf{t}_k; d\mathbf{v}_k). \tag{3.7}$$

PROOF. Let $z_j(A, \delta)$ be defined analogously for μ_j, so that for arbitrary fixed m and $0 < \varepsilon \leqq \delta$, $Z_{m,\varepsilon}(A, \delta) = \sum_{j=0}^{N_{m,\varepsilon}} z_j(A, \delta)$. For $\mu_j = \mu_{\mathbf{t}_k, \mathbf{v}_k}$

$$\int \log \xi(t, v) z_j(dt, dv) = \sum_{i=1}^k \log \xi(t_i, v_i).$$

As in Theorem 3.4,

$$\begin{aligned} &E_{m,\varepsilon} \exp\left[\int_{R_+ \times T} \log \xi(t, v) z_j(dt, dv)\right] \\ &\qquad = L_{m,\varepsilon}^{-1} \sum_{k=1}^{\infty} \frac{1}{k!} \int_{[\varepsilon,\infty)^k} \int_{T_m^k} \prod_{i=1}^k \xi(t_i, v_i) \Lambda_k(d\mathbf{t}_k; d\mathbf{v}_k), \end{aligned}$$

whence $Z_{m,\varepsilon}$ has log probability generating functional

$$\sum_{k=1}^{\infty} \frac{1}{k!} \int_{[\varepsilon,\infty)^k} \int_{T_m^k} \prod_{i=1}^k [\xi(t_i, v_i) - 1] \Lambda_k(d\mathbf{t}_k; d\mathbf{v}_k).$$

Since $Z_{m,\varepsilon} \to Z(w)$, as $m \to \infty$, $\varepsilon \downarrow 0$, Z must have probability generating functional (3.7). □

COROLLARY 3.6. *If X has probability generating functional G_∞, then:* (i) *for fixed $\delta > 0$ the stochastic point process $Z(\cdot, \delta)$ has probability generating functional G_δ given by*

$$\log G_\delta[\xi] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{T^k} \prod_{i=1}^k [\xi(t_i) - 1] Q_k(d\mathbf{t}_k; [\delta, \infty)^k);$$

and (ii) *for fixed* $A \in \mathscr{T}$, *the stochastic point process* $Z(A, \cdot)$ *has probability generating functional* G_A *given by*

$$\log G_A[\xi] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{R_+^k} \prod_{i=1}^{k} [\xi(v_i) - 1] Q_k(A^k; d\mathbf{v}_k) .$$

PROOF. To show (i), apply Theorem 3.5 with $\log \xi(t, v) = I_{[\delta,\infty)}(v) \log \xi(t)$, while (ii) follows by letting $\log \xi(t, v) = I_A(t) \log \xi(v)$. □

$Z(\cdot, \delta)$ and $Z(A, \cdot)$ thus each have probability generating functionals of the form (2.8), i.e., are i.d. stochastic point processes.

With the above structure established, an integral representation for X in terms of Z is now given.

THEOREM 3.7. *If* Y *is a random measure with probability generating functional* G_∞, *then the integral*

$$I(\cdot) = \int_{R_+} vZ(\cdot, dv) \quad \text{a.s.}$$

has probability generating functional G_∞.

PROOF. From the form (3.7) of the probability generating functional of Z it follows that

$$G_\infty[\xi] = E \exp\left[\int_{R_+ \times T} \log \xi^v(t) Z(dt, dv)\right] . \tag{3.8}$$

By Fubini's theorem, (3.8) equals

$$E \exp\left[\int_T \log \xi(t) \int_{R_+} vZ(dt, dv)\right] ,$$

which implies the desired result. □

Theorem 3.7 generalizes the Lévy–Itô representation for processes with independent increments to random measures with probability generating functionals of the form G_∞. This theorem shows that such random measures are purely atomic, so that if each $Q_k(dt; \cdot)$ is concentrated on k-tuples of nonnegative integers it follows that X is a stochastic point process. Notice that while $Z(A, 0)$ is at most countably infinite, $Z(A, \delta)$ is a.s. finite for all compact A, i.e., it is a stochastic point process. The following theorem gives a weak finiteness condition on $Z(A, 0)$.

THEOREM 3.8. *For* X *having probability generating functional* G_∞, $P[Z(A, 0) < \infty] > 0$ *for all compact* A iff

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k!} Q_k(A^k; {R_+}^k) < \infty . \tag{3.9}$$

PROOF. By Corollary 3.6(i), $Z(\cdot, 0)$ has probability generating functional

$$\log G[\xi] = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{T^k} \prod_{i=1}^{k} [\xi(t_i) - 1] Q_k(d\mathbf{t}_k; {R_+}^k) .$$

Hence $P[Z(A, 0) < \infty] > 0$ iff

$$\begin{aligned} -\infty &< \log E \exp(-uZ(A, 0)) \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{m=1}^{k} \binom{k}{m} (e^{-mu} - 1) \Lambda_k(A^m \times (T \backslash A)^{k-m}; {R_+}^k) , \end{aligned} \tag{3.10}$$

and (3.10) holds iff

$$(3.11) \qquad \textstyle\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{m=1}^{k} \binom{k}{m} \Lambda_k(A^m \times (T\backslash A)^{k-m}; R^{+k}) < \infty .$$

From the definition of Λ_k, (3.11) may be written as (3.9). □

The family of stochastic processes defined on a topological vector space T (usually taken to be R) having probability generating functional (3.1) can be shown to contain the class of Poisson cluster stochastic point processes by introducing a more general class of Poisson cluster processes. Suppose that cluster centers occur in T according to a nonstationary Poisson process $N(\cdot)$ having intensity function $I(\cdot)$. A cluster of atoms is associated with and distributed about each center, and it is assumed w.l.o.g. that an atom occurs at the cluster center. Furthermore, clusters are assumed to be i.i.d. and independent of $N(\cdot)$. A cluster ψ_s centered at s contains k atoms w.p. $\Pi_k(s)$. Given s, the respective magnitudes $v_1, \cdots, v_k$ and locations $s, s+u_1, \cdots, s+u_{k-1}$ have a joint distribution $W_k(d\mathbf{u}_{k-1}; d\mathbf{v}_k \mid s)$ on $T^{k-1} \times R_+{}^k$.

The cluster ψ_s then has probability generating functional

$$(3.12) \qquad G_{\psi_s}[\xi] = \textstyle\sum_{k=1}^{\infty} \Pi_k(s) \int_{T^{k-1}\times R_+^k} \prod_{i=1}^{k} \xi^{v_i}(s+u_{i-1}) W_k(d\mathbf{u}_{k-1}; d\mathbf{v}_k \mid s)$$

where $T^0 = \emptyset$ and $u_0 = 0$, and the process has probability generating functional

$$(3.13) \qquad \log G[\xi] = \textstyle\int_T [G_{\psi_s}[\xi] - 1] I(ds) .$$

See Daley and Vere-Jones ((1974), (5.2.4)). Since each $\Pi_\bullet(s)$ and $W_k(\cdot \mid s)$ is a probability distribution, (3.12) and (3.13) imply

$$\log G[\xi] = \textstyle\sum_{k=1}^{\infty} \int_{T^k \times R_+^k} [\prod_{i=1}^{k} \xi^{v_i}(s+u_{i-1}) - 1] W_k(d\mathbf{u}_{k-1}; d\mathbf{v}_k \mid s) \Pi_k(s) I(ds) .$$

This functional is of the form G_∞ with $t_{i+1} = s + u_i$, $0 \leqq i \leqq k-1$, $k \geqq 1$, and

$$\Lambda_k(d\mathbf{t}_k; d\mathbf{v}_k) = k!\, W_k(d\mathbf{u}_{k-1}; d\mathbf{v}_k \mid s) \Pi_k(s) I(ds) .$$

It follows that $Z(\cdot, \partial)$ and $Z(A, \cdot)$ are each Poisson cluster point processes, so that the class $\mathscr{D}$ may be considered as the natural generalization of the class of Poisson cluster stochastic point processes.

The random measures considered here thus far have all been elements of $\mathscr{D}$, i.e., those with KLM measures concentrated on D. In practice, however, one may be interested in random measures with limited aftereffects, in the sense that events occurring far apart in time ($T = R$) should be "almost" independent. Rather than deal with mixing properties or ergodicity, however, dependency across time is described in terms of the finite dimensional distributions of the process.

Denote by G_n the probability generating functional G_∞ for which $Q_k \equiv 0$ for all $k > n$.

THEOREM 3.9. *An* i.d. *random measure X has probability generating functional of the form G_n* iff $X \in \mathscr{D}_n$.

PROOF. The result is immediate from Theorem 3.1. $\square$

In this case, X is the limit of Poisson sums of clusters having no more than n atoms.

Theorem 3.10 gives yet a further characterization of the elements of $\mathscr{D}_n$, expressed this time in terms of their finite dimensional distributions. Although it is obtained through somewhat more laborious calculations, the authors found it somewhat more intriguing than Theorem 3.9.

Define, for arbitrary disjoint bounded $A_1, \cdots, A_m \in \mathscr{T}$, $u_1, \cdots, u_m \in R_+$ and $m \geqq 1$, the *h-function* of X by

$$h(\mathbf{u}_m; \mathbf{A}_m) = -\log E \exp[-\textstyle\sum_{i=1}^m u_i X(A_i)] = -\log G[\eta_m] \,. \tag{3.14}$$

(See Lee (1967)). Next, define for $p \leqq m$ the sum

$$\phi_m(p) = \textstyle\sum_{1 \leqq i_1 < \cdots < i_p \leqq m} h(u_{i_1}, \cdots, u_{i_p}; A_{i_1}, \cdots, A_{i_p}) \,. \tag{3.15}$$

THEOREM 3.10. *For fixed* $n \geqq 1$, $X \in \mathscr{D}_n$ iff *there exists a probability generating functional of the form* G_n *such that the h-functions of* X *satisfy*

(i) $$h(\mathbf{u}_m; \mathbf{A}_m) = -\log G_n(\eta_m) \,, \qquad 1 \leqq m \leqq n \,,$$

(ii) $$h(\mathbf{u}_m; \mathbf{A}_m) = \textstyle\sum_{p=1}^n (-1)^{n-p} \binom{m-p-1}{n-p} \phi_m(p) \,, \qquad m > n \,.$$

PROOF. For each r, $1 \leqq r \leqq n$, and $\mathbf{a}_r = (a_1, \cdots, a_r)$ such that $1 \leqq a_j \leqq m$, $1 \leqq j \leqq r$, define

$$x_r(\mathbf{a}_r) = -\frac{1}{r!} \int_{R_+^r} \textstyle\prod_{i=1}^r [\exp(-u_{a_i} v_i) - 1] Q_r(A_{a_1} \times \cdots \times A_{a_r}; d\mathbf{v}_r) \,. \tag{3.16}$$

If $X \in \mathscr{D}_n$, then X has probability generating functional of the form G_n and

$$\begin{aligned} h(\mathbf{u}_m; \mathbf{A}_m) &= \textstyle\sum_{r=1}^n \sum_{a_1=1}^m \cdots \sum_{a_r=1}^m x_r(\mathbf{a}_r) \\ &= \textstyle\sum_{r=1}^n \sum_{k=1}^r \sum_{1 \leqq a_1 < \cdots < a_k \leqq m} \sum_{b_1 + \cdots + b_k = r; (b_i > 0)} \binom{r}{\mathbf{b}_k} \\ &\qquad \times x_r(a_1^{(b_1)}, \cdots, a_k^{(b_k)}) \,, \end{aligned} \tag{3.17}$$

where $a^{(b)}$ denotes the subscript a repeated b times in the argument of x_r. It can be seen from (3.16) and (3.17) that for $m > n$

$$\begin{aligned} \phi_m(p) = \textstyle\sum_{j=1}^p \binom{m-j}{p-j} \sum_{1 \leqq a_1 < \cdots < a_j \leqq m} \sum_{r=j}^n \sum_{b_1 + \cdots + b_j = r; (b_i > 0)} \binom{r}{\mathbf{b}_j} \\ \times x_r(a_1^{(b_1)}, \cdots, a_j^{(b_j)}) \,. \end{aligned} \tag{3.18}$$

The equality

$$\textstyle\sum_{r=0}^i (-1)^r \binom{j}{r} = (-1)^i \binom{j-1}{i} \,, \qquad j > i \geqq 0 \,,$$

and (3.18) together imply

$$\begin{aligned} \textstyle\sum_{p=1}^n (-1)^{n-p} \binom{m-p-1}{n-p} \phi_m(p) &= \textstyle\sum_{k=1}^n \sum_{p=1}^k (-1)^{k-p} \binom{m-p}{k-p} \phi_m(p) \\ &= \textstyle\sum_{k=1}^n \sum_{r=k}^n \sum_{1 \leqq a_1 < \cdots < a_k \leqq m} \sum_{b_1 + \cdots + b_k = r; (b_i > 0)} \binom{r}{\mathbf{b}_k} \\ &\qquad \times x_r(a_1^{(b_1)}, \cdots, a_k^{(b_k)}) \\ &= h(\mathbf{u}_m; \mathbf{A}_m) \,, \qquad m > n \,. \end{aligned} \tag{3.19}$$

Conversely, if there exists a probability generating functional of the form G_n such that (i) and (ii) hold, then from (3.18)

$$h(\mathbf{u}_m; \mathbf{A}_m) = -\log G_n[\eta_m], \qquad m \geqq 1 .$$

Thus the probability generating functional of X must agree with G_n of all η_m, hence for all $\xi \in V_0$, and so G_n is the probability generating functional of X. Theorem 3.9 implies that $X \in \mathscr{D}_n$. □

Since the h-function is the negative logarithm of the m.g.f. of $\mathbf{X}(\mathbf{A}_m)$, all finite dimensional distributions of $X \in \mathscr{D}_n$ are determined by the finite dimensional distributions of dimension $\leqq n$. Of particular interest here is the case in which X is a stochastic point process in $\mathscr{D}$. In this case X is a Poisson cluster process having $\leqq n$ points in each cluster (see Ammann and Thall (1977)) iff all finite dimensional distributions are determined by the finite dimensional distributions of dimensions $\leqq n$ in the manner specified by Theorem 3.10.

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