Stochastic Processes and their Applications 16 (1983) 205–210
North-Holland

SHORT COMMUNICATION

A THEOREM ON REGULAR INFINITELY DIVISIBLE COX PROCESSES

Peter F. THALL
Department of Statistics, George Washington University, Washington, D.C. 20052, USA

Received 1 February 1981
Revised 21 January 1982

If a regular infinitely divisible (Poisson cluster) point process is Coxian (doubly stochastic Poisson, subordinated Poisson), then the number of points per cluster either takes on each positive integer value with positive probability or is identically equal to one. In particular, a Gauss–Poisson process can not be Coxian.

point process
Cox process
Poisson process
infinitely divisible point process
Laplace functional

1. Introduction and summary

Let ξ be a regular infinitely divisible (Poisson cluster) point process, and denote by κ the (random) number of points in a given cluster. It is the purpose of this note to show that if ξ is also a Cox (doubly stochastic Poisson, subordinated Poisson) process, then κ either takes on each positive integer value with positive probability or is degenerate at one. The essential importance of this result lies in its implications for modeling, insofar as each of these general classes presently enjoys rather wide usage. The literature on cluster processes is extensive and well known, and the family of Cox processes is the limiting class for a broad variety of special cases, as shown most generally in [5]. Given the prominent role of Cox and in particular mixed Poisson processes in stochastic geometry and point process theory, the theorem is of interest in that it shows what a Cox process can *not* be. It follows from our Theorem, for example, that the doubly stochastic Poisson process suggested as a possible example of a Gauss–Poisson process, per satisfaction of certain admissibility conditions, by Milne and Westcott [10, pp. 170, 171] does not exist.

A Cox process ξ may be thought of as a Poisson process having an intensity which is itself stochastic, in consequence of ξ evolving in a continuum subject to some externally motivated random effect. See, for example, Kingman [6], Mecke [9], Krickeberg [7], Kallenberg [4, 5]. A Poisson cluster process, equivalently a regular infinitely divisible (i.d.) point process (cf. Matthes, Kerstan and Mecke [8, Chapter 4]), may be constructed by superposing a sequence of independent clusters of points, each centered around or generated by one point of a 'parent' Poisson process. See [1, 2, 3, 8, 10], among many others. For both models, the degenerate case is that of the Poisson process where, respectively, $\kappa = 1$ a.s. for the Poisson cluster process and the intensity is nonrandom for the Cox process. The Theorem below asserts that, aside from the degenerate case, a given point process cannot arise via both of these mechanisms working at once unless κ is unbounded. This formally includes the singular i.d. case where $P[\kappa = \infty] > 0$, although this class is not treated here per se.

I.d. Cox processes have been considered by several authors. The present article was motivated in part by a note of Shanbhag and Westcott [11], who also cite some earlier work. Kallenberg [5, Exercise 8.6] gives an example of an i.d. Coxian variable having a directing variable which is not i.d. Cox and Isham [3, pp. 80, 81] give an example of a regular i.d. Cox process ξ having rate function $\Lambda(t) = \int_0^t w(t-u)\xi_c(du)$, where ξ_c is a Poisson process and w a suitable weight function. In this case ξ is a Neyman–Scott process with $P[\kappa = k] > 0$ for all positive integer k.

The proof of the Theorem exploits characteristic representations for the respective Laplace functionals of the Cox and regular i.d. processes, in conjunction with standard Laplace functional inversion theory. The argument also relies on Mecke's elegant characterization of the Cox process as a point process which is, for any $p \in (0, 1]$, a p-thinning of some point process.

2. Notation and preliminaries

Let the state space S be a locally compact second countable Hausdorff space endowed with the ring $\mathcal{B}$ of bounded Borel subsets. Denote by $\mathcal{M}$ the set of all locally finite (Radon) measures μ on $(S, \mathcal{B})$, with $\mathcal{B}_{\mathcal{M}}$ the smallest sigma-algebra over $\mathcal{M}$ for which all mappings $\mu \to \mu B$, $B \in \mathcal{B}$, are measurable. A *random measure* (r.m.) η is any measurable mapping from a probability space into $(\mathcal{M}, \mathcal{B}_{\mathcal{M}})$; equivalently η is any random element of $\mathcal{M}$. Denote $\mathbb{Z} = \{0, 1, 2, \ldots\}$, $\mathbb{Z}_+ = \mathbb{Z} - \{0\}$, and the zero element of $\mathcal{M}$ by 0. A *stochastic point process* (spp) is any a.s. $\mathcal{N}$-valued r.m., where $\mathcal{N} = \{\mu \in \mathcal{M} : \mu B \in Z, B \in \mathcal{B}\}$ and we define $\mathcal{B}_{\mathcal{N}} = \mathcal{B}_{\mathcal{M}} \cap \mathcal{N}$. Let $\mathcal{F}$ be the set of nonnegative real-valued measurable functions on S which have compact support, and for convenience denote $\mu f = \int_S f \, d\mu$, $f \in \mathcal{F}$, $\mu \in \mathcal{M}$. The *Laplace functional* $\mathcal{L}_\eta$ of a r.m. η is defined by $\mathcal{L}_\eta(f) = E(e^{-\eta f})$, $f \in \mathcal{F}$, and the *Laplace transform* L_X of an $R_+ = [0, \infty)$-valued r.v. X by $L_X(\theta) = E(e^{-\theta X})$, $\theta > 0$. Thus $\mathcal{L}_\eta(\theta 1_B) = L_{\eta B}(\theta)$, where 1_B is the indicator function of the set B. For any $\mu \in \mathcal{M}$, denote by Π_μ the probability distribution of a Poisson process ξ having mean measure (intensity) μ; $\mathscr{L}_\xi(f)=\exp(-\mu(1-e^{-f}))$ in this case, and we write $\xi \stackrel{d}{=} \Pi_\mu$. By allowing μ to be replaced by the r.m. η having distribution V, the distribution $P_V = \int_{\mathscr{M}} \Pi_\mu V(\mathrm{d}\mu)$ uniquely defines the *Cox process directed by* η. In this case, $\mathscr{L}_\xi(f) = \mathscr{L}_\eta(1-e^{-f})$, and we write $\xi \stackrel{d}{=} C(\eta)$.

The fact that the distribution of any r.m. ($\mathbb{R}_+$-valued r.v.) is characterized by its Laplace functional (transform) will be used implicitly in all that follows. A detailed treatment of Laplace functionals, as well as proofs of our Lemmas 1 and 2 below, can be found in Kallenberg [4].

Lemma 1 (Mecke). *An spp* $\xi \stackrel{d}{=} C(\eta)$ *if and only if, for every* $p \in (0,1]$, ξ *is a p-thinning of some spp* $\xi^{(p)}$. *In this case,* $\xi^{(p)} \stackrel{d}{=} C(p^{-1}\eta)$.

Lemma 2. *An spp* ξ *is infinitely divisible if and only if*

$$\log(\mathscr{L}_\xi(f)) = \int_{\mathscr{N}} [e^{-\mu f}-1]\lambda(\mathrm{d}\mu), \quad f \in \mathscr{F}, \tag{2.1}$$

where λ *is the unique measure on* $(\mathscr{N}-\{0\}, \mathscr{B}_{\mathscr{N}})$ *satisfying* $\lambda\{\mu B>0\}<\infty$, $B \in \mathscr{B}$.

An i.d. spp is *regular* if and only if λ is concentrated on $\{\mu \in \mathscr{N}: \mu S < \infty\}$ and *singular* if and only if λ is concentrated on $\{\mu \in \mathscr{N}: \mu S = \infty\}$.

Lemma 3. (Ammann and Thall). *The following three statements are equivalent:*

ξ is a regular i.d. spp;

$$\log(\mathscr{L}_\xi(f)) = \sum_{k=1}^{\infty} \int_{S^k} \left(\exp\left(-\sum_{j=1}^{k} f(t_j)\right)-1\right) \Lambda_k(\mathrm{d}\mathbf{t}_k), \quad f \in \mathscr{F}, \tag{2.3}$$

where Λ_k *is a symmetric Radon measure on* $(S^k, \mathscr{B}^k)$, $k \in \mathbb{Z}_+$, *such that*

$$0 \leq \sum_{k=1}^{\infty} \sum_{r=k}^{\infty} \binom{r}{k} \Lambda_r(B^k \times S^{r-k}) < \infty, \quad B \in \mathscr{B};$$

$$\log(\mathscr{L}_\xi(f)) = \sum_{k=1}^{\infty} \int_{S^k} \prod_{j=1}^{k} [e^{-f(t_j)}-1] Q_k(\mathrm{d}t_k), \quad f \in \mathscr{F}, \tag{2.4}$$

where Q_k *is a symmetric Radon measure on* $(S^k, \mathscr{B}^k)$, $k \in \mathbb{Z}_+$, *such that*

$$0 \leq \sum_{k=1}^{\infty} (-1)^{k-1} Q_k(B^k) < \infty, \quad B \in \mathscr{B}.$$

The equivalent representations of $\mathscr{L}_\xi$ given in (2.3) and (2.4) are related by the formulae

$$Q_k(B) = \sum_{m=0}^{\infty} \binom{m+k}{k} \Lambda_{m+k}(B \times S^m), \tag{2.5.1}$$

$$\Lambda_k(B)=\sum_{m=0}^{\infty}\binom{m+k}{k}(-1)^m Q_{m+k}(B\times S^m),\quad B\in\mathscr{B}^k,\ k\in\mathbb{Z}_+. \tag{2.5.2}$$

The representation (2.3) is obtained from that given in (2.1) via the identification $\Lambda_k(\mathrm{d}t_k)=k!\,\lambda\{\delta_{t_1}+\cdots+\delta_{t_k}\}$, where $\delta_t B=1_B(t)$. Intuitively, Λ_k is the (transformed) component of λ accounting for clusters which contain k unit atoms, referred to hereafter as 'k-clusters'. Since only the regular case is considered here, there are no '∞-clusters'. Any of the Λ_k's in (2.3) may be multiplied by corresponding nonnegative constants without destroying the validity of the representation, provided that the existence condition still holds. However, if some Λ_m appears in the expression for $\mathscr{L}_\xi$ given in (2.3), or equivalently if the process has m-clusters, then all of the Q_k's, $1\leq k\leq m$, must appear in the alternate expression for $\mathscr{L}_\xi$ given in (2.4). For brevity, denote by $\mathscr{D}_m$ the class of regular i.d. spps for which $\Lambda_m\not\equiv 0$ but $\Lambda_k\equiv 0$ for all $k>m$, i.e. those processes having a maximum of m unit atoms in each cluster. Let $\mathscr{D}_\infty$ denote those regular i.d. spps for which $\Lambda_k\not\equiv 0$, $k\in Z_+$, i.e. clusters of all sizes occur with positive probability.

3. Regular i.d. Cox processes

Theorem. *If an spp ξ is both Coxian and regular i.d. then either $\xi\in\mathscr{D}_1$ or $\xi\in\mathscr{D}_\infty$.*

Proof. Assume first that $\xi\in\mathscr{D}_m$ for some integer $m\geq 2$. By Lemma 1, for each $p\in(0,1]$ there exists an spp $\xi^{(p)}$ such that ξ is a p-thinning of $\xi^{(p)}$. Fix p. The Laplace functional of ξ takes the general form given by (2.4), with m in place of ∞, and moreover

$$\mathscr{L}_\xi(f)=\mathscr{L}_{\xi^{(p)}}(-\log[1-p(1-e^{-f})]),\quad f\in\mathscr{F}. \tag{3.1}$$

Upon setting $\theta h=-\log[1-p(1-e^{-f})]$ for $\theta>0$ and inverting (3.1), since $\xi\in\mathscr{D}_m$ it follows that

$$L_{\xi^{(p)}h}(\theta)=\mathscr{L}_{\xi^{(p)}}(\theta h)=\exp\left[\sum_{k=1}^{m}\int_{S^k}\prod_{j=1}^{k}(e^{-\theta h(t_j)}-1)p^{-k}Q_k(\mathrm{d}\mathbf{t}_k)\right], \tag{3.2}$$

for $0\leq\theta<\|h\|^{-1}(-\log(1-p))$, $h\in\mathscr{F}$. By analytic continuation, temporarily regarding θ as an element of the complex plane, we have this expression for $L_{\xi^{(p)}h}$ valid for all $\theta>0$, and hence it is valid for $\mathscr{L}_{\xi^{(p)}}$ for all $h\in\mathscr{F}$.

Regarding ξ as a $C(\eta)$ process, since $\mathscr{L}_\xi(f)=\mathscr{L}_\eta(1-e^{-f})$ we may likewise invert this equation (cf. [5, p. 17]) to obtain

$$L_{\eta B}(\theta)=\mathscr{L}_\eta(\theta 1_B)=e^{-\psi(\theta,B)},\quad B\in\mathscr{B},\ 0\leq\theta<1, \tag{3.3}$$

where $\psi(\theta,B)=\sum_{k=1}^{m}(-1)^{k-1}\theta^k Q_k(B^k)$. Again by analytic continuation, expression (3.3) holds for all $\theta>0$.

If $\psi(\theta, B)<0$ for some $\theta>0$ and $B \in \mathscr{B}$, then $L_{\eta B}(\theta)>1$, which is impossible. Writing $p=\theta^{-1}$ for $\theta \geqslant 1$, we see that the condition in (2.4) of Lemma 3 is satisfied, so that $\xi^{(p)} \in \mathscr{D}_m$ for all $p \in (0, 1]$, with $p^{-k}Q_k$ in place of Q_k and m in place of ∞ in the expression for $\mathscr{L}_\xi$. By the nature of the thinning mechanism, ξ must have k-clusters with positive probability, equivalently $\Lambda_k \not\equiv 0$, for all $k=1, \ldots, m$. Denoting by $\{\Lambda_k^{(p)}\}_{k=1}^m$ the measures appearing in the alternate form (2.3) of the Laplace functional of $\xi^{(p)}$, it follows from equation (3.1) that

$$\Lambda_k = p^k \sum_{r=0}^{m-k} \binom{r+k}{k} (1-p)^r \Lambda_{r+k}^{(p)}(\cdot \times S^r), \quad 1 \leqslant k \leqslant m. \tag{3.4}$$

Upon evaluating this expression for $k=m-1$ and $k=m$ and solving for $\Lambda_{m-1}^{(p)}$, it follows that, for any $B \in \mathscr{B}^{m-1}$,

$$\Lambda_{m-1}^{(p)}(B)<0 \quad \text{if} \quad p<(m\Lambda_m(B \times S))/(m\Lambda_m(B \times S)+\Lambda_{m-1}(B)).$$

Since this contradicts the existence conditions for $\xi^{(p)}$ given in Lemma 3, it must be the case that $m=1$ or $m=\infty$.

4. Complements and extensions

The following Corollary is an immediate consequence of the formal inversion of $\mathscr{L}_\xi$ performed in the proof of the Theorem.

Corollary. *Suppose that ξ is a p-thinning of an spp $\xi^{(p)}$ for some $p \in (0, 1]$. Then $\xi \in \mathscr{D}_m$ if and only if $\xi^{(p)} \in \mathscr{D}_m$, for each $m \in Z_+ \bigcup \{\infty\}$.*

Given an spp $\xi \in \mathscr{D}_m$ and $p \in (0, 1]$, Proposition 1.13.3 of [8] ensures the existence of a signed r.m. $\xi^{(p)}$ of which ξ is a p-thinning, and it is easily seen that $\xi^{(p)}$ is purely atomic. However, $\xi^{(p)}$ is not necessarily an spp, unless $m=1$ or $m=\infty$, and for $m \in Z_+ - \{1\}$ if $\xi^{(p)}$ is an spp then it must be in $\mathscr{D}_m$. In any case, $\mathscr{L}_{(p)}$ takes the form given in (2.3) with

$$\Lambda_k^{(p)} = \sum_{r=0}^{m-k} \binom{r+k}{k} (-q)^r p^{-(r+k)} \Lambda_{r+k}(\cdot \times S^r) \tag{4.1}$$

in place of Λ_k for each k.

The proof that no element of $\mathscr{D}_m$ can be Coxian is somewhat easier for m even. Here a contradiction can be obtained by deriving $L_{\eta B}(\theta)$ as before and then choosing

$$\theta > \max_{1 \leqslant k \leqslant m/2} \{Q_{2k-1}(B^{2k-1})/Q_{2k}(B^{2k})\},$$

which implies that $L_{\mu B}(\theta)>1$. In particular, this is a quick proof that the class of Gauss–Poisson processes ($\mathscr{D}_2$ in our notation) is disjoint from the class of Cox processes.

The Theorem says essentially that randomizing the intensity of a Poisson process is phenomenologically compatible with clustering, provided that the spp has clusters of all sizes. Given this restriction, it seems reasonable to consider a process in some $\mathscr{D}_m$ which is subject to certain random effects due to the environment in which it evolves. This might be formulated by independently endowing each point of a Cox process with a cluster, or more generally by considering the entire spp to be subject to the externally generated random effects. To this end, define a probability distribution V on the set of all canonical measures λ which satisfy the conditions of Lemma 2, along with the appropriate specification of an algebra of measurable sets of such measures. A generalized Cox process would then have a distribution defined by the mixture $\int \Xi_\lambda V(\mathrm{d}\lambda)$, where Ξ_λ denotes the probability law of an element of $\mathscr{D}_m$ having canonical measure λ. This construction is valid provided that $\mathscr{L}_\xi(f)$ is measurable in λ for all $f \in \mathscr{F}$, by Lemma 1.7 of [5]. For this process, mixing λ is equivalent to randomizing $\{\Lambda_k\}_{k=1}^m$, which is in turn equivalent to randomizing the intensity, distribution of κ and spatial distribution of the clusters, obtaining a doubly stochastic Poisson cluster process.

Acknowledgment

The author wishes to thank Bob Smythe for his thorough reading of an earlier draft of this paper.

References

[1] L.P. Ammann and P.F. Thall, On the structure of regular infinitely divisible point processes, Stochastic Process. Appl. 6 (1977) 87–94.
[2] L.P. Ammann and P.F. Thall, Count distributions, orderliness and invariance of Poisson cluster processes, J. Appl. Probab. 16 (1979) 261–273.
[3] D.R. Cox and V. Isham, Point Processes (Chapman and Hall, London and New York, 1980).
[4] O. Kallenberg, Limits of compound and thinned point processes, J. Appl. Probab. 12 (1975) 269–278.
[5] O. Kallenberg, Random Measures (Akademie-Verlag, Berlin, 1976).
[6] J.F.C. Kingman, On doubly stochastic Poisson processes, Proc. Camb. Phil Soc. 60 (1964) 923–930.
[7] K. Krickeberg, The Cox process, Inst. Nat. Alta Mat., Symp. Math. 9 (1972) 151–167.
[8] K. Matthes, J. Kerstan and J. Mecke, Infinitely Divisible Point Processes (Wiley, New York, 1978).
[9] J. Mecke, Eine charakteristische Eigenschaft der doppelt stochastischen Poissonschen prozesse, Z. Wahrsch. und verw. Gebiete 11 (1968) 74–81.
[10] R.K. Milne and M. Westcott, Further results for the Gauss–Poisson process, Adv. Appl. Probability 4 (1972) 151–176.
[11] D.N. Shanbhag and M. Westcott, A note on infinitely divisible point processes, J. Roy. Statist. Soc. Ser. B 39 (1977) 331–332.