Wavelet-Based Functional Mixed Models

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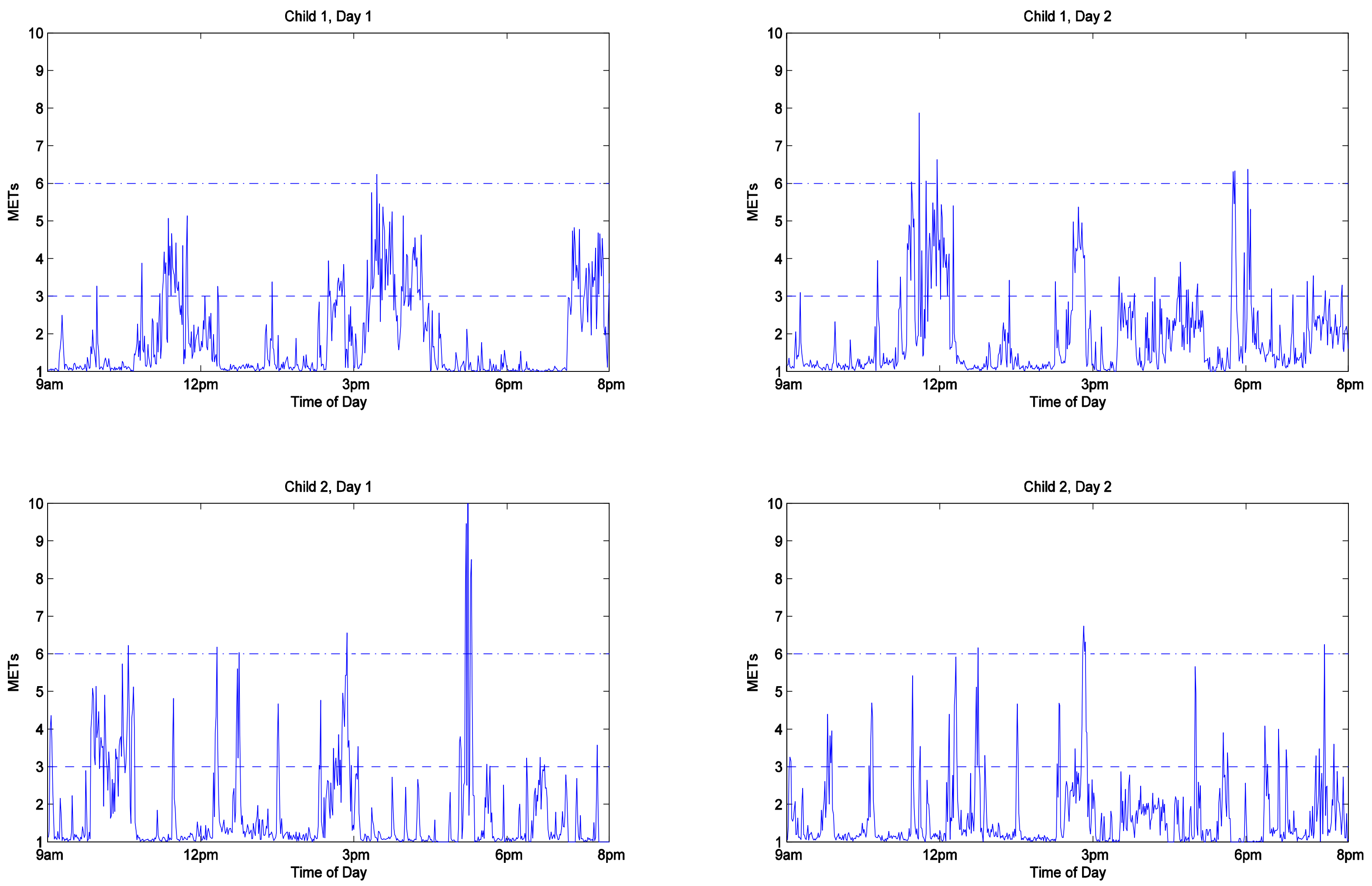
Functional Data

- **Functional Data**:
  - Ideal units of observation: curves
  - Observed data: curves sampled on fine grid
- Increasingly encountered in biomedical research because of new technologies taking automated measurements
- Present unique challenges:
  - Extremely large data sets (>100s-1000s per curve)
  - Curves may have complex, irregular, non-smooth structure

Example: Accelerometer Data

- **Accelerometers:** small motion sensors that digitally record minute-by-minute activity levels
  - Increasingly used in surveillance and intervention studies
- **TriTrac-R3D:** sensor worn on hip
  - Minute-by-minute record of motion in 3 planes
  - Condensed into single activity level measurement/minute
  - Activity "profile" for each day

Accelerometer Data



Accelerometer Data

- **Planet Health Study (Harvard University):**
  - **Intervention study investigating activity levels of middle school children in Boston area schools**
  - **Children's activity levels objectively monitored using TriTrac-R3D activity monitor for one or two 4-day sessions**
  - **Data considered: 292 daily profiles/103 children/5 schools *660 measurements/profile* (every minute from 9am-8pm)**
- **Goals:**
  1. **Assess how activity levels vary throughout day, across schools, across different days of the week, over time from early to late Spring, and across various child-level covariates.**
  2. **Assess relative variability in activity levels from day-to-day and child-to-child, in order to guide future study design.**

Linear Mixed Models

Linear Mixed Model (Laird and Ware, 1982):

$$\underbrace{Y}_{N\times 1} = \underbrace{X}_{N\times p}\overbrace{\beta}^{p\times 1} + \underbrace{Z}_{N\times m}\overbrace{u}^{m\times 1} + \underbrace{e}_{N\times 1}$$

$$\boldsymbol{u} \sim \boldsymbol{N}(\boldsymbol{0}, \overbrace{\boldsymbol{D}}^{\boldsymbol{m}\times\boldsymbol{m}})$$
$$\boldsymbol{e} \sim \boldsymbol{N}(\boldsymbol{0}, \underbrace{\boldsymbol{R}}_{\boldsymbol{N}\times\boldsymbol{N}})$$

- **Fixed effects part, $X\beta$, accommodate a broad class of mean structures, including main effects, interactions, and linear coefficients.**
- **Random effects part, Zu, provide a convenient mechanism for modeling correlation among the N observations.**

Functional Mixed Model

Suppose we observe a sample of N curves, $Y_i(t)$, i=1, …, N, all defined on $\mathcal{T}$

$$U_k(t) \sim GP(0, Q)$$

$$E_i(t) \sim GP(0, S)$$

$$Y_i(t) = \sum_{j=1}^{p} X_{ij} B_j(t) + \sum_{k=1}^{m} Z_{ik} U_k(t) + E_i(t)$$

- $B_j(t)$ **= fixed effect functions**
- $U_k(t)$ **= random effect functions**
- $E_i(t)$ **= residual error processes**
- **Q and S are covariance surfaces on $\mathcal{T} \times \mathcal{T}$**
  - $S(t_1,t_2)$ **= Cov{$E_i(t_1),E_i(t_2)$}: describes within-curve covariance structure of residual curve-to-curve deviations**

Functional Mixed Models

- **Key feature of FMM**: Does not require specification of parametric form for curves
- Kernels/fixed-knot splines may not work well for spatially heterogeneous data – inherent smoothness assumptions attenuate local features
- **Wavelet Regression**: nonparametric regression technique that better preserves local features present in the curves.

Introduction to Wavelets

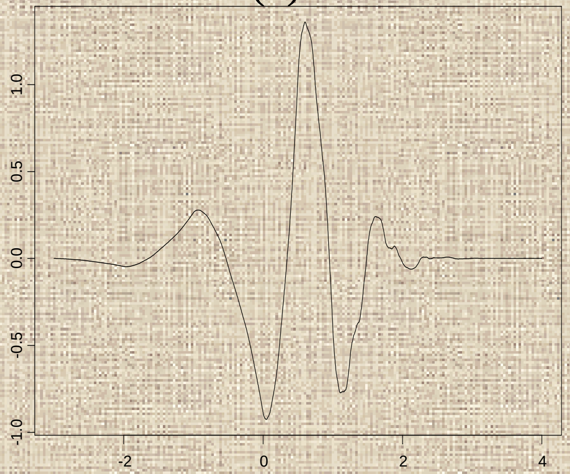
- **Wavelets: families of orthonormal basis functions**

$$g(t) = \sum_{j,k \in \mathfrak{I}} d_{jk} \psi_{jk}(t)$$

$$\psi_{jk}(t) = 2^{-j/2} \psi(2^{-j/2} t - k)$$

$$d_{jk} = \int g(t) \psi_{jk}(t) dt$$

Daubechies (4) Basis Function

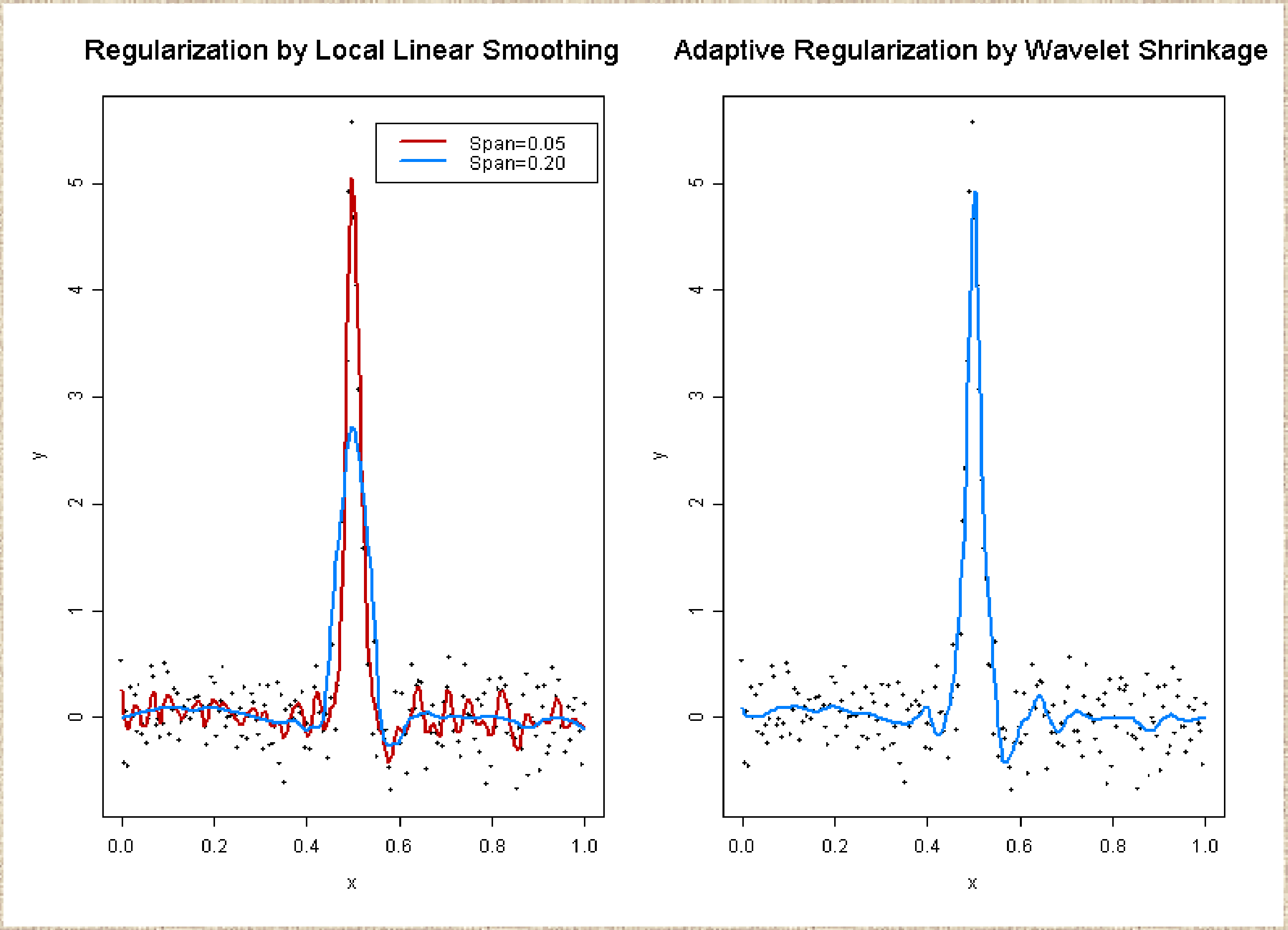


- **Discrete Wavelet Transform (DWT): fast algorithm {O(T)} for obtaining T empirical wavelet coefficients for curves sampled on equally-spaced grid of length T.**
- **Linear Representation:** $d = y W'$
  - W' = T-by-T orthogonal projection matrix
- **Inverse DWT (IDWT):** $y = d W$

Wavelet Regression

- **Wavelet Regression – 3 step process**
  1. **Project data into wavelet space**
  2. **Threshold/shrink coefficients**
  3. **Project back to data space**
- **Yields *adaptively regularized* (plot) nonparametric estimates of function**
- **Morris, et al. (2003) extended to hierarchical functional model (Bayesian)**
- **Morris and Carroll (2004) extended to general functional mixed model framework (wavelet-based functional mixed model)**

Adaptive Regularization



Functional Mixed Model
(Discrete version)

Y= *N*-by-*T* matrix containing the observed spectra on sampling grid of size *T*

$$\underbrace{Y}_{N\times \mathrm{T}} = \overbrace{X}^{N\times p}\underbrace{B}_{p\times \mathrm{T}} + \overbrace{Z}^{N\times m}\underbrace{U}_{m\times \mathrm{T}} + \underbrace{E}_{N\times \mathrm{T}}$$

$$U_i \sim MVN(0,Q)$$
$$E_i \sim MVN(0,S)$$

- B_{ij} **is the effect of covariate *i* at location** t_j
- ***Q*** **and *S* are covariance matrices (*T* x *T*)**
- **Note: Some structure must be assumed on form of Q and S (discussed later)**

Wavelet-Based FMM:

General Approach

1. **Project observed functions Y into wavelet space.**
2. **Fit FMM in wavelet space.**

   (Use MCMC to get posterior samples)
3. **Project wavelet-space estimates (posterior samples) back to data space.**

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Wavelet-Based FMM

1. **Project observed functions Y to wavelet space**

- **Apply DWT to rows of Y to get wavelet coefficients corresponding to each observed function**

$$\underbrace{D}_{N\times T} = \underbrace{Y}_{N\times T} \underbrace{W'}_{T\times T}$$

- **Projects the observed curves into the space spanned by the wavelet bases.**

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Projecting FMM to Wavelet Space

$$\underbrace{Y}_{N\times \mathrm{T}} = \overbrace{X}^{N\times p} \underbrace{B}_{p\times \mathrm{T}} + \overbrace{Z}^{N\times m} \underbrace{U}_{m\times \mathrm{T}} + \underbrace{E}_{N\times \mathrm{T}}$$

$$U_i \sim MVN(0,Q)$$

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Projecting FMM to Wavelet Space

$$\underbrace{Y}_{N\times \mathrm{T}} \overbrace{\mathbf{W'}}^{T\times T} = \overbrace{X}^{N\times p} \underbrace{B}_{p\times \mathrm{T}} + \overbrace{Z}^{N\times m} \underbrace{U}_{m\times \mathrm{T}} + \underbrace{E}_{N\times \mathrm{T}}$$

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$$U_i \sim MVN(0,Q)$$

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Projecting FMM to Wavelet Space

$$\underbrace{Y}_{N\times\mathrm{T}}\overbrace{\mathbf{W'}}^{\mathbf{T}\times\mathbf{T}} = \overbrace{X}^{N\times p}\underbrace{B}_{p\times\mathrm{T}}\overbrace{\mathbf{W'}}^{T\times T} + \overbrace{Z}^{N\times m}\underbrace{U}_{m\times\mathrm{T}}\overbrace{\mathbf{W'}}^{T\times T} + \underbrace{E}_{N\times\mathrm{T}}\overbrace{\mathbf{W'}}^{T\times T}$$

$$U_i\mathbf{W'} \sim MVN(0, \mathbf{W}Q\mathbf{W'})$$

$$E_i\mathbf{W'} \sim MVN(0, \mathbf{W}S\mathbf{W'})$$

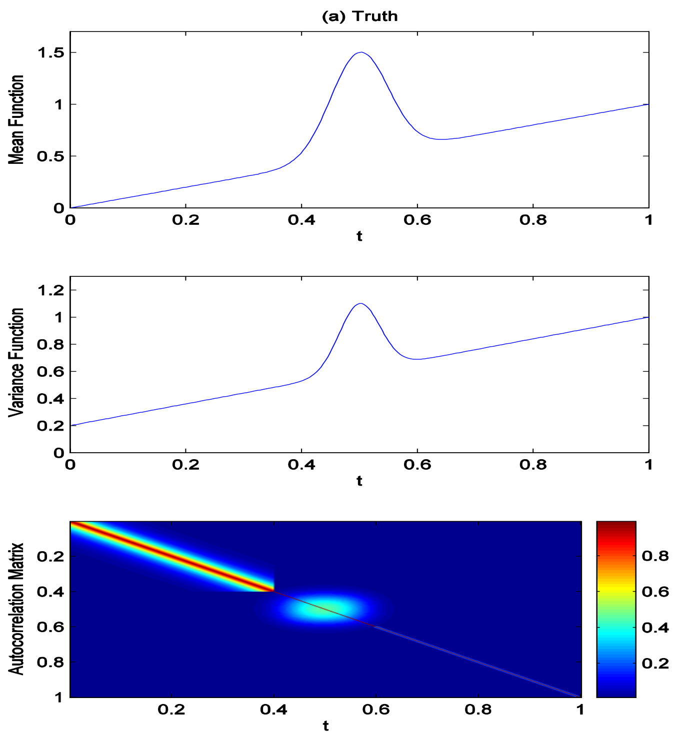
Projecting FMM to Wavelet Space

$$\underbrace{\mathbf{D}}_{N\times \mathrm{T}} = \overbrace{X}^{N\times p}\underbrace{\mathbf{B}^*}_{p\times \mathrm{T}} + \overbrace{Z}^{N\times m}\underbrace{\mathbf{U}^*}_{m\times \mathrm{T}} + \underbrace{\mathbf{E}^*}_{N\times \mathrm{T}}$$

$$\mathbf{U}_i^* \sim MVN(0, \mathbf{Q}^*)$$

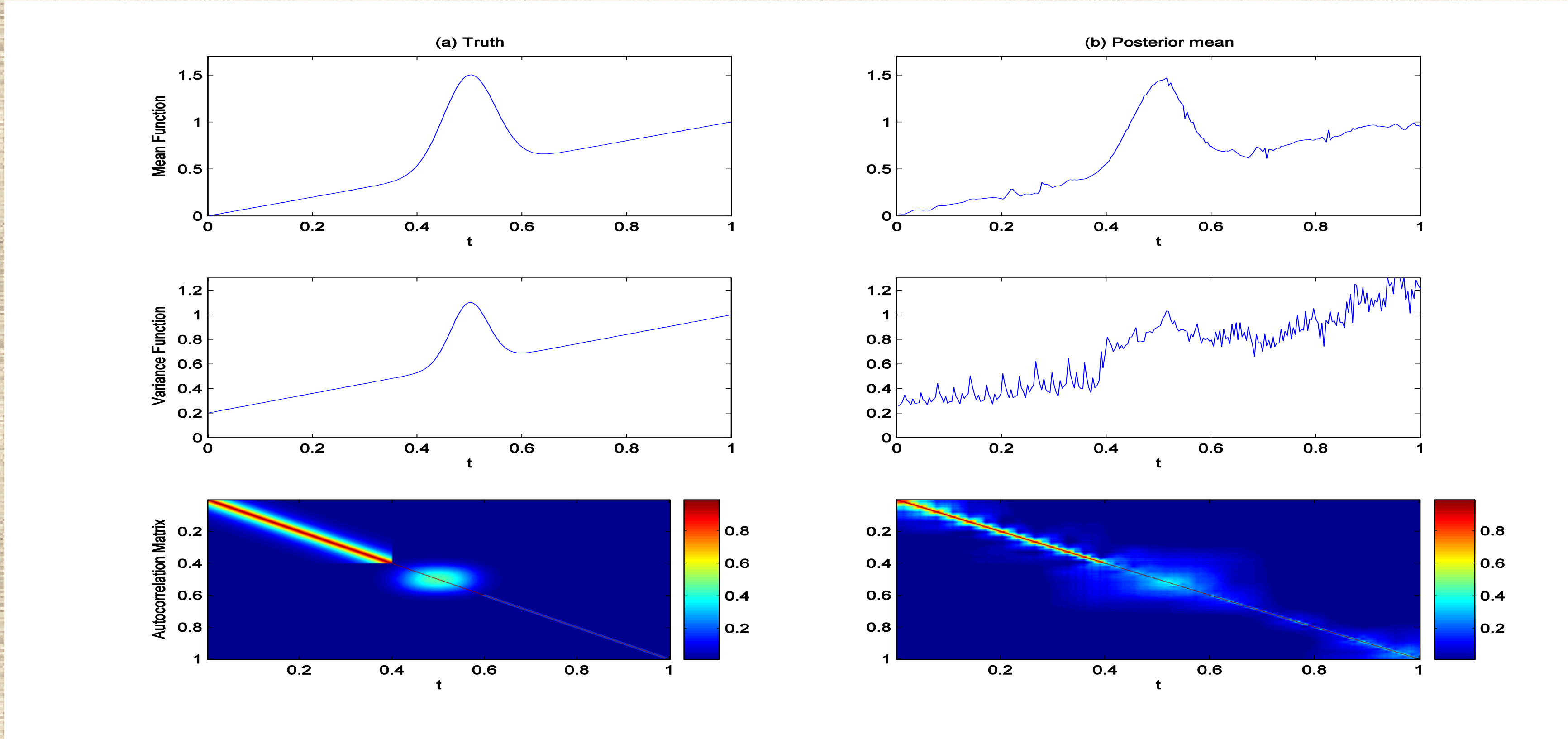
$$\mathbf{E}_i^* \sim MVN(0, \mathbf{S}^*)$$

Simulation: Covariance Structure



- **True mean**: line plus peak
- **True variance:** increasing in t, with extra var at peak
- **True autocorrelation:** Strong autocorrelation (0.9) at left, weak autocorrelation (0.1) right, extra at peak

Simulation: Covariance Structure



- **Independence in wavelet space** **accommodates varying degrees of** **autocorrelation in data space**
- **Allowing variance components to vary across scale j and location k accommodates** **nonstationarities**

Model Each Column Separately

$$\underbrace{d_{jk}}_{N\times 1} = \overbrace{X}^{N\times p} \underbrace{\beta^*_{jk}}_{p\times 1} + \overbrace{Z}^{N\times m} \underbrace{u^*_{jk}}_{m\times 1} + \underbrace{e^*_{jk}}_{N\times 1}$$

$$\boldsymbol{u^*_{jk} \sim N(0, q^*_{jk})}$$

$$\boldsymbol{e^*_{jk} \sim N(0, s^*_{jk})}$$

Prior Assumptions

Mixture prior on β_{ijk}^*:

$$\beta_{ijk}^* = \gamma_{ijk}^* N(0, \tau_{ij}) + (1 - \gamma_{ijk}^*)\delta_0$$

$$\gamma_{ijk}^* = \text{Bernoulli}(\pi_{ij})$$

- **Nonlinearly shrinks β_{ijk}^* towards 0, leading to adaptively regularized estimates of $\beta_i(t)$.**
- **τ_{ij} & π_{ij} are regularization parameters**
  - **Can be estimated from the data using empirical Bayes**
  - **Extend Clyde&George (1999) to functional mixed model**

Model Fitting

- **MCMC to obtain posterior samples of model quantities**
  - **Work with marginal likelihood; U* integrated out;**
- **Let Ω be a vector containing ALL covariance parameters (i.e. Q^* and S^*).**

MCMC Steps

1. **Sample from $f(B^*|D,\Omega)$:**

   Mixture of normals and point masses at 0 for each i,j,k.

2. **Sample from $f(\Omega|D,B^*)$:**

   Metropolis-Hastings steps for each j,k

3. **If desired, sample from $f(U^*|D,B^*,\Omega)$:**

   Multivariate normals

Wavelet-Based FMM:

General Approach

1. **Project observed functions Y into wavelet space.**
2. **Fit FMM in wavelet space**
   (Use MCMC to get posterior samples)
3. **Project wavelet-space estimates (posterior samples) back to data space.**

Wavelet-Based FMM

3. **Project wavelet-space estimates (posterior samples) back to data space.**

- **Apply IDWT to posterior samples of *B** to get posterior samples of fixed effect functions $B_j(t)$ for $i=1,\ldots, p$, on grid t.**
  - **B=B*W**
- **Posterior samples of $U_k(t)$, Q, and S are also available, if desired.**
- **Can be used for Bayesian inference/prediction**

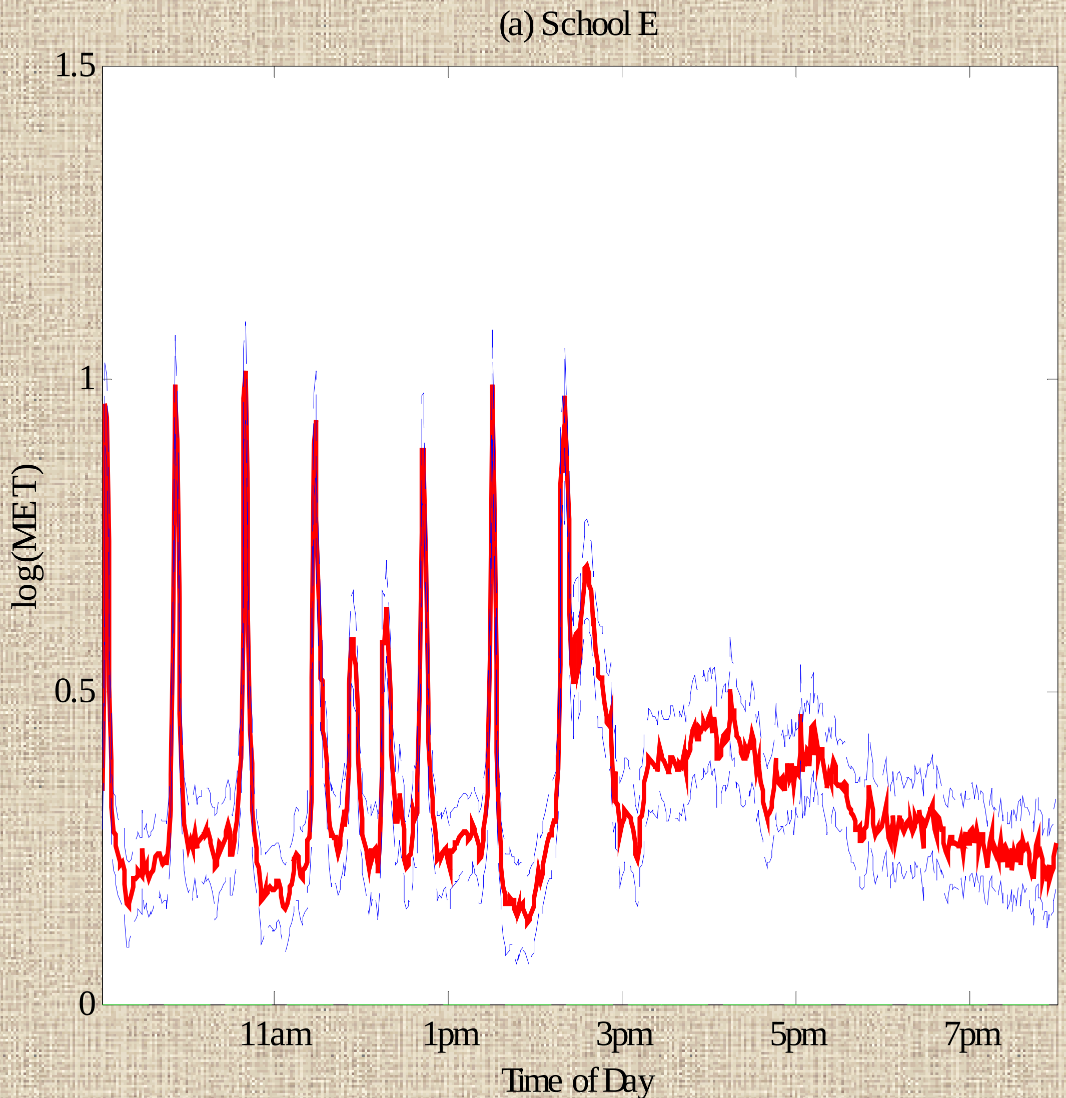
Model

Let Y be 292×660 matrix containing *292* accelerometer profiles for each minute from 9am-8pm.

$$\boxed{Y = XB + ZU + E}$$

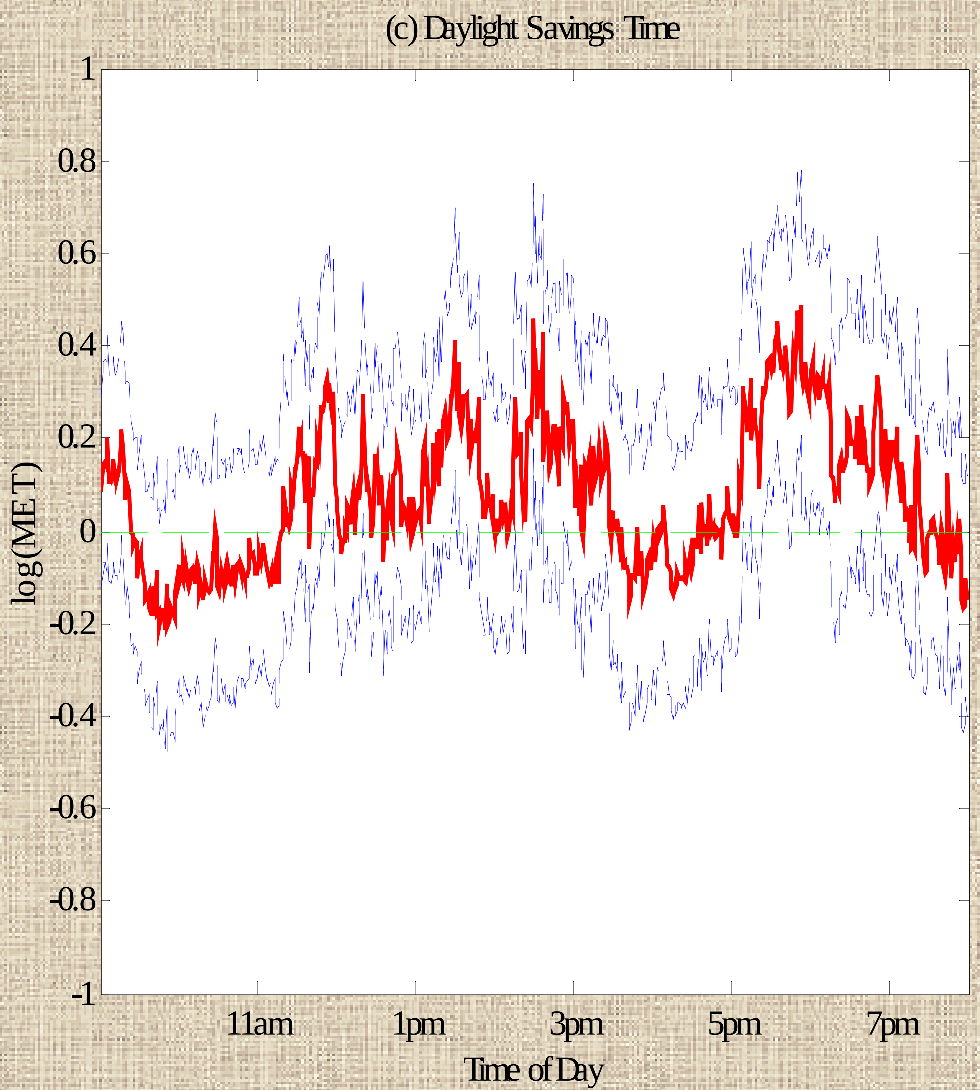
- **X = 292 × 14 matrix of covariates**
  - **School effects (5), gender, % body fat, BMI, day-of-week (4), daylight savings time, avg tv hrs/wk**
- **B = 14 × 660 matrix of fixed effects functions**
  - **B_{ij} is effect of covariate i at time t_j**
- **Z = 292 × 106 matrix indicating child for each profile U = 106 × 660 matrix of random effect functions (1/child)**
- **E = 292 × 660 matrix of residual errors**

Selected Results: School Effects



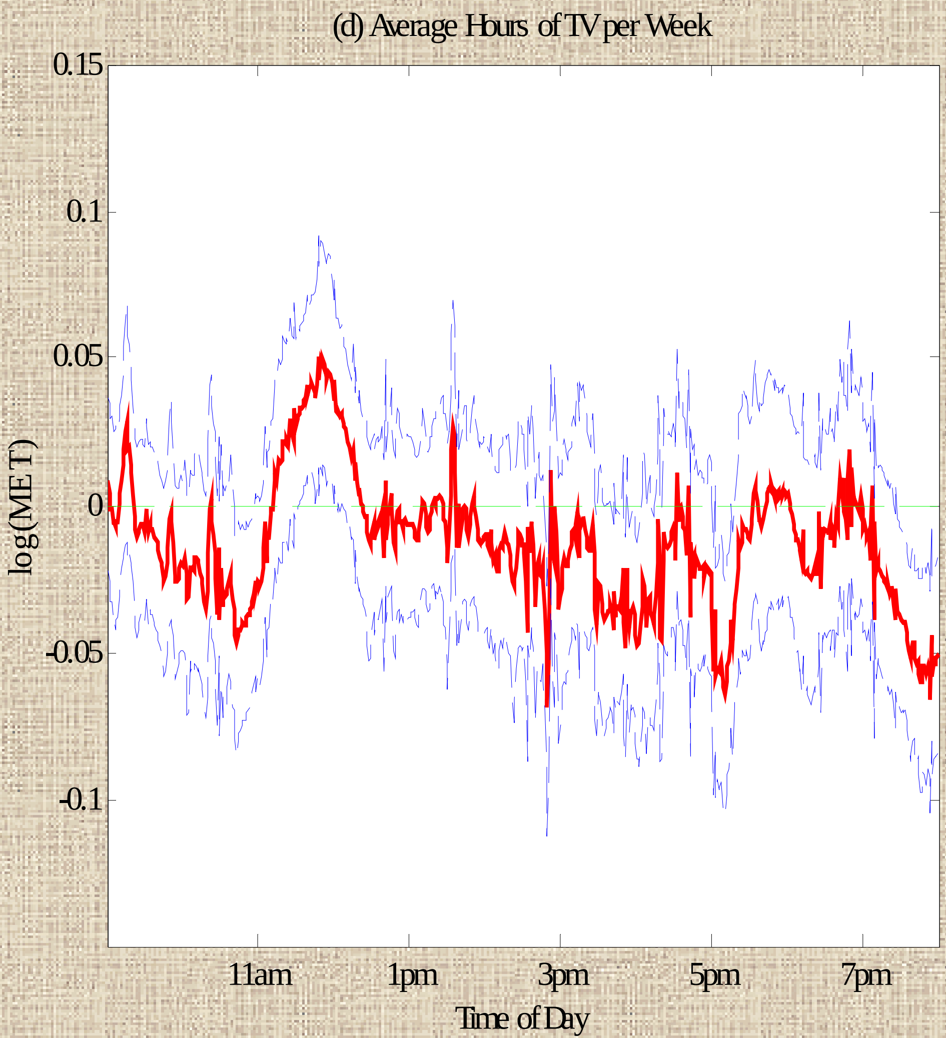
- **School schedules evident in effects**
  - Spikes every 48min (changing classes)
  - 3 lunch periods
  - School out at 2:15pm
- **Not so evident in individual curves**

Selected Results: DST Effect



- **DST – April 6th**
- **More active after DST (overall 8%, *p=0.062*)**
- **Especially strong:**
  - **As school is letting out (2:15-3:00, 25%, *p=0.03*)**
  - **In early evening (5:30-7:00, 30%, *p=0.01*)**
- **Note: Sunset was**
  - **5:10-6:15 before DST**
  - **7:15-8:10 after DST**

Selected Results: TV hours/wk

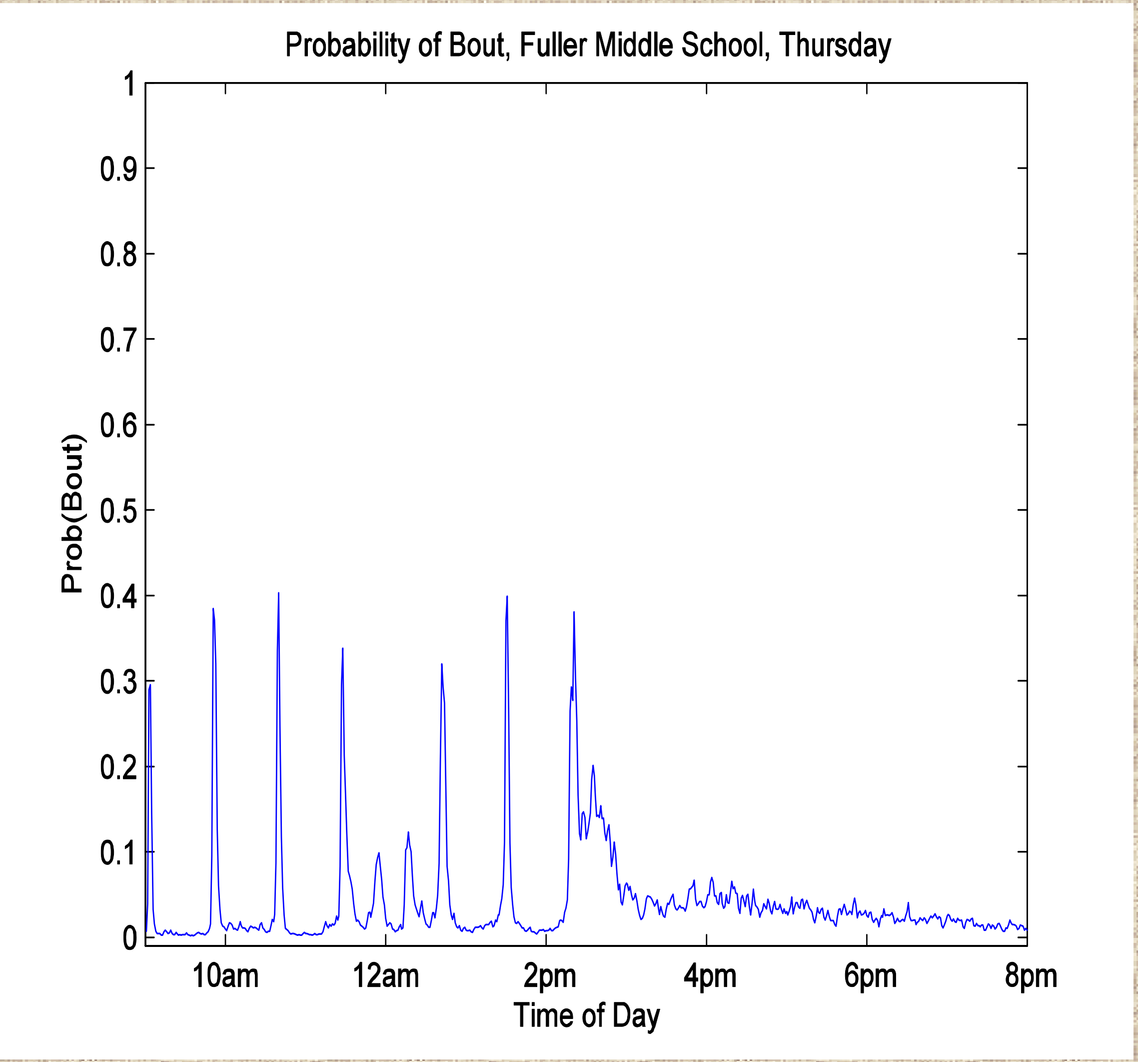


- **TVhrs coded as continuous factor (standardized)**
- **TVhrs effect negative (-1.3% per sd, *p=0.03*)**
  - **More TV, less active**
  - **3:00-5:30, -2.6%, *p=0.02***
  - **7:00-8:00, -3.6%, *p=0.008***
- **Positive effect over lunch**
  - **+2.7%, *p=0.03***
  - **More TV, on average more active over lunch**

Results: Covariance Analysis

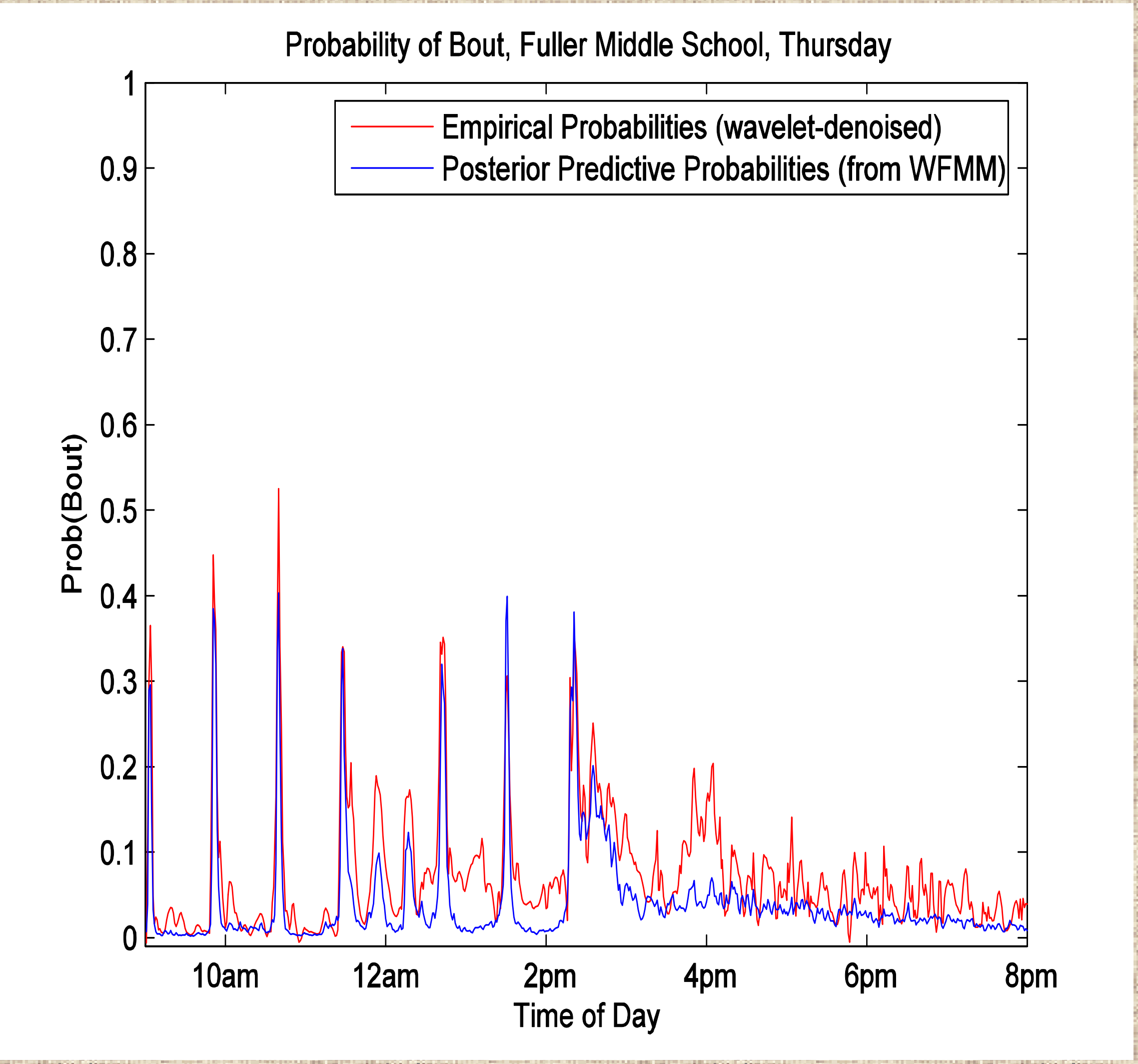
- **Variability: 91% day-to-day, 9% child-to-child**
  - **Important to have many days per child**
- **Study variability as function of *t***
  - **Child-to-child variability: school day > after school**
  - **Day-to-day variability: after school > school day**
  - **Relative day-to-day variability after school: 95%-99%**
- **Equivalent designs:**
  - **108 children, 4 days/child**
  - **72 children, 8 days/child**
  - **54 children, 16 days/child**
- **Less children, more days, save $$$?**

Results: Bouts



- **Can compute posterior predictive probabilities of bouts for children**

Results: Bouts



- Can compute posterior predictive probabilities of bouts for children
- Model-based predictive probabilites not far from empirically-estimated probabilities
- May want heavier tails

Discussion

- **Introduced unified modeling approach for FDA**
  - **Can accommodate very irregular functions**
- **Method based on mixed models; is** FLEXIBLE
  - **Accommodates a wide range of experimental designs**
  - **Addresses large number of research questions**
- **Posterior samples allow Bayesian inference and prediction**
  - **Posterior credible intervals; pointwise or joint**
  - **Predictive distributions for future sampled curves**
  - **Predictive probabilities for classification of new curves**
  - **Bayesian functional inference can be done via Bayes Factors**
- **Since a unified modeling approach is used, all sources of variability in the model propagated throughout inference.**

Discussion

- Approach is Bayesian. The only informative priors to elicit are regularization parameters, which can be estimated from data using empirical Bayes.
- Developed general-use code – reasonably fast and straightforward to use → minimum information to specify is Y, X, Z matrices.
- Can deal with missing data, i.e. partially observed functions (not discussed here)
- Method generalizes to higher dimensional functions, e.g. image data, space/time (fixed domain) data.

Acknowledgements

- **Work presented here is from 2 papers (both currently under review)**

1. **"*Wavelet-Based Functional Mixed Models*" (2004) Jeffrey S. Morris and Raymond J. Carroll**
2. **"*Using Wavelet-Based Functional Mixed Models to Characterize Population Heterogeneity in Accelerometer Profiles: A Case Study*" (2005) Jeffrey S. Morris, Cassandra Arroyo, Brent Coull, Louise Ryan, Richard Herrick, and Steve Gortmaker**